The Periodicity Tower

Modular Sequence Periods Across the Octave Representation

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1 Introduction

A companion paper [1] introduced the *octave representation* of the positive integers: for a positive integer n and a base $b \geq 3$, the pair

$$\mathrm{note}(n, b) = 1 + ((n-1) \bmod (b-1)), \qquad \mathrm{octave}(n, b) = \lceil n/(b-1) \rceil$$

together with the iterated *tower* $L_k(n) = \mathrm{note}(O^{k-1}(n), b)$, where O is the octave map $n \mapsto \mathrm{octave}(n, b)$. The main result of [1] (the Tower Theorem) was that the tower is precisely the sequence of base-$(b-1)$ digits of $n-1$, with the digit alphabet shifted by $+1$. As a corollary — the Dependence Law — every level $L_k(n)$ depends only on $n \bmod (b-1)^k$.

The Dependence Law is the engine of this paper. It says, immediately, that any integer sequence whose values are periodic modulo $(b-1)^k$ will have its level-k signal periodic too, with period dividing the modular period. That observation is one half of a quantitative law about how integer sequences appear inside the octave representation. The other half is the question of *which* sequences make the divides relation tight — for which the level-k period equals the modular period of the underlying sequence — and where the relation is strict, and what the strictness reveals.

This paper develops the periodicity story for four sequence families: the harmonic series, the Fibonacci numbers, the Lucas numbers, and the polygonal numbers, together with the primes as the natural control. The four are chosen because each makes a different aspect of the modular story visible.

- The **harmonic series** is the simplest case: an arithmetic progression with common difference f. Its level-k signal period divides $(b-1)^k/\gcd(f, (b-1)^k)$, with equality provable at level 1 and observed without exception in computational testing at every higher level. No surprises; the example is included as the cleanest demonstration that the framework recovers what arithmetic intuition predicts.
- The **Fibonacci numbers** are the deep case. Their periods modulo any integer are the classical Pisano periods $\pi(d)$; at level k, the Fibonacci tower signal has period dividing $\pi((b-1)^k)$. We call the resulting law the *Pisano tower*. Whether the relation is an equality remains open in general. We prove the divides direction here, state the equality as a conjecture, and report it as confirmed empirically up to level $k = 3$.

- The **Lucas numbers** carry an anomaly that the Fibonacci numbers do not: the level-1 period equals the Pisano period $\pi(b-1)$, *except* when 5 divides $b-1$, in which case it collapses to $\pi(b-1)/5$. The factor of 5 is the discriminant of the Fibonacci–Lucas characteristic polynomial $x^2 - x - 1$, and the anomaly is the first signature, inside the octave representation, of a structural fact about linear recurrences: discriminants matter. We document the anomaly here and state its higher-level generalization as an open question.
- The **polygonal numbers** are quadratic in their index, and their level-1 periods exhibit a clean dichotomy: either $b-1$ or $2(b-1)$, depending on the parity of the generating quadratic.
- The **primes** are aperiodic modulo every integer greater than two (Dirichlet equidistribution), and so their tower signal is aperiodic at every level and every base. This was the control case in [1]; we restate it for completeness and quantify it slightly further.

Together, these five families exhibit Theorem 4 (the Tower Periodicity Theorem, stated in §2) as operational rather than abstract. They also exhibit the choice of base b as a parameter that selects which arithmetic structure of a sequence becomes legible — a thread carried over from [1] and developed by example here.

The Lucas anomaly will turn out to be a particular instance of a more general discriminant-driven phenomenon. The general statement, and the connection to a family of classical questions about linear recurrences, does not belong to the modest scope of this paper; we record the anomaly at level 1 and signal that its generalization remains open.

1.1 Structure

Section 2 recalls the necessary definitions and notation from [1], states Theorem 4 (the Tower Periodicity Theorem), and proves it. The proof is a single line from the Dependence Law; the theorem's interest is in its specializations. Section 3 treats the harmonic series and proves the exact gcd-tower formula. Section 4 treats Fibonacci and proves the Pisano-tower divisibility. Section 5 treats Lucas, develops the discriminant-driven anomaly at level 1, and states the higher-level open question. Section 6 treats polygonal numbers and restates prime aperiodicity. Section 7 closes with scope, related work, and the open questions.

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2 Setup and the Tower Periodicity Theorem

We recall the definitions and the single corollary from [1] on which this paper rests, then state and prove the umbrella periodicity theorem.

2.1 Recall from the Companion Paper

Fix an integer base $b \geq 3$. For a positive integer n, define

$$\mathrm{note}(n, b) = 1 + ((n-1) \bmod (b-1)), \qquad \mathrm{octave}(n, b) = \lceil n/(b-1) \rceil.$$

The constraint $b \geq 3$ is essential and excludes the degenerate case $b = 2$. Let $O\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ denote the octave map $n \mapsto \mathrm{octave}(n, b)$; its iterates descend to the fixed point $O(1) = 1$ in finitely many steps. The *tower* of n is the sequence

$$L_k(n) = \mathrm{note}(O^{k-1}(n), b) \qquad \text{for } k = 1, 2, 3, \ldots,$$

where the dependence on b is suppressed in the notation but always implicit. We write $m = n - 1$ throughout; the shift is what makes the structure transparent.

The Tower Theorem of [1] identifies $L_k(n) - 1$ with the k-th least-significant base-$(b-1)$ digit of m. Its corollary supplies the engine of the present paper:

> **Corollary 2a (Dependence Law) [1].** For every $n, k \geq 1$ and $b \geq 3$, $L_k(n)$ depends only on $n \bmod (b-1)^k$.

This says that the level-k value of the tower is determined entirely by the residue of n modulo $(b-1)^k$. If two integers share the same such residue, they share the same level-k value. Periodicity propagates immediately.

2.2 The Tower Periodicity Theorem

> **Theorem 4 (Tower Periodicity).** Let $(a_j)_{j\geq 1}$ be a sequence of positive integers, and suppose (a_j) is periodic modulo $(b-1)^k$ with period P — that is, $a_{j+P} \equiv a_j \pmod{(b-1)^k}$ for all $j \geq 1$. Then the level-k signal $(L_k(a_j))_{j\geq 1}$ is periodic with period dividing P.

Proof. By Corollary 2a, $L_k(a_j)$ is a function of the residue $a_j \bmod (b-1)^k$. If a_{j+P} and a_j share that residue, their level-k values agree: $L_k(a_{j+P}) = L_k(a_j)$. The level-k signal therefore repeats with period dividing P. □

A short proof. The interest of the theorem is not its statement but the question it makes precise: for *which* integer sequences is the divides relation an equality? The remainder of the paper takes up that question for four sequence families.

2.3 What the Theorem Leaves Open

The divides relation in Theorem 4 is unidirectional: a modular period of (a_j) at modulus $(b-1)^k$ supplies an upper bound on the level-k signal's period, but the signal's true period may be strictly smaller. Whether the divides relation is tight — whether the level-k signal period *equals* the modular period of the underlying sequence — is the family-by-family question this paper takes up.

The shape of the answer differs by family.

- For the **harmonic series** (§3), the divides relation is tight at level 1 by a direct calculation; equality at higher levels is observed without exception in computational testing but not proven.
- For the **Fibonacci numbers** (§4), the divides direction is provable and equality is conjectured but open. We will prove $\text{period}_k(F) \mid \pi((b-1)^k)$ — the Pisano-tower divisibility law — and quote computational evidence for equality up to $k = 3$.
- For the **Lucas numbers** (§5), the divides direction is again provable, but the bound is not always tight at level 1; the gap is exactly the discriminant-driven anomaly.
- For the **polygonal numbers** (§6), the divides direction yields the $(b-1)$ vs $2(b-1)$ dichotomy.
- For the **primes** (§6), no modular period exists in the first place, and Theorem 4 has no content to apply. The signal is aperiodic at every level.

Theorem 4 is a one-line corollary of Corollary 2a from [1], and so the entire periodicity story is, structurally, the modular-periodicity story of base-$(b-1)$ digits. The specializations that follow are not standalone arithmetic results but applications of one classical fact — that the digits of a periodic-mod-d^k sequence in base d are themselves periodic — under a representation change.

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3 The Harmonic Series

The simplest periodic integer sequence is the *harmonic series* with fundamental f:

$$(a_j)_{j\geq 1} = f,\ 2f,\ 3f,\ 4f,\ \ldots$$

For any fixed $f \geq 1$, the harmonic series is periodic modulo every positive integer d, with period equal to $d/\gcd(f,d)$ — the smallest positive integer T such that $Tf \equiv 0 \pmod{d}$. Applied at $d = (b-1)^k$:

$$T_k(f,b) \ := \ \frac{(b-1)^k}{\gcd(f,(b-1)^k)}$$

is the modular period of $(jf)_j$ at modulus $(b-1)^k$.

Theorem 5 (Harmonic Tower Period). For all $f \geq 1$, $b \geq 3$, and $k \geq 1$, the level-k signal $\big(L_k(jf)\big)_{j\geq 1}$ is periodic with period dividing

$$T_k(f,b) \ = \ \frac{(b-1)^k}{\gcd(f,(b-1)^k)}.$$

At $k = 1$, equality holds: the period is exactly $T_1(f,b) = (b-1)/\gcd(f,b-1)$.

Proof. The divides direction is Theorem 4 applied to the modular period $T_k(f,b)$ of $(jf)_j$ at modulus $(b-1)^k$.

For the equality at $k = 1$: by Lemma 1 of [1], $L_1(n) - 1 = (n-1) \bmod (b-1)$, so $L_1(jf) - 1 = (jf-1) \bmod (b-1)$. The map from residues $\bmod (b-1)$ to the alphabet $\{0,1,\ldots,b-2\}$ is a bijection, so the period of $(L_1(jf))_j$ equals the period of $((jf-1) \bmod (b-1))_j$, which is exactly $T_1(f,b)$. $\square$

A small worked example: take $f = 2$, $b = 5$, $k = 2$, giving $T_2(2,5) = 16/\gcd(2,16) = 8$. Computing $L_2(2j)$ for $j = 1,2,\ldots,9$ yields the sequence $1,1,2,2,3,3,4,4,1$ — period exactly 8, matching the formula.

The equality direction at higher levels $k \geq 2$ does not follow from a bijection of the same kind: the map $n \mapsto L_k(n)$ is not injective on residues modulo $(b-1)^k$, since multiple residues can share the same k-th base-$(b-1)$ digit. Nevertheless, computational testing of equality across many hundreds of triples (f,b,k) found no exceptions. We conjecture the equality holds in general and leave a proof open.[1]

The simplicity of this case sets a baseline. The Fibonacci numbers, treated in §4, have periods modulo $(b-1)^k$ that are *not* easily computed from gcds — they are the Pisano periods, which require their own arithmetic — but the divides direction supplied by Theorem 4 applies uniformly, and the equality question becomes the deeper open problem.

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[1] In computational testing across triples (f,b,k) spanning $f \leq 3$, $b \leq 50$, and $k \leq 3$, the level-k signal period was computed directly and compared to the formula; no discrepancies arose.

4 Fibonacci Numbers and the Pisano Tower

The Fibonacci numbers $(F_j)_{j\geq 1} = 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$ are defined by $F_1 = F_2 = 1$ and $F_{j+2} = F_{j+1} + F_j$. They are periodic modulo every positive integer d, with period $\pi(d)$ — the *Pisano period* of d, named after Leonardo of Pisa (Fibonacci). The Pisano period has been studied since at least Lagrange (1774); it satisfies a multiplicative law $\pi(mn) = \operatorname{lcm}(\pi(m), \pi(n))$ when $\gcd(m, n) = 1$, and its values at small d are tabulated in standard references.

For prime moduli, $\pi(p)$ has a sharp closed form in terms of the Legendre symbol $(5/p)$: $\pi(p)$ divides $p - 1$ if 5 is a quadratic residue modulo p, and divides $2(p + 1)$ if it is not (Wall, 1960). For prime powers, an empirical formula holds:

$$\pi(p^k) \;=\; p^{k-1} \cdot \pi(p)$$

with no known exceptions — no prime p has ever been observed for which $\pi(p^2) = \pi(p)$ rather than $p \cdot \pi(p)$. Such a prime, were it to exist, would be called a *Wall–Sun–Sun prime*, and its existence remains a celebrated open question. We will return to this point in §4.3.

4.1 The Pisano Tower

Applied at the modulus $(b-1)^k$, the Pisano period $\pi((b-1)^k)$ is the period of $(F_j)_j$ modulo $(b-1)^k$. Theorem 4 immediately yields the divisibility:

> **Theorem 6 (Pisano Tower — Divisibility).** For all $b \geq 3$ and $k \geq 1$, the level-k Fibonacci signal $\big(L_k(F_j)\big)_{j\geq 1}$ is periodic with period dividing $\pi((b-1)^k)$.

Proof. The Fibonacci sequence is periodic modulo $(b-1)^k$ with period $\pi((b-1)^k)$. By Theorem 4, the level-k signal is periodic with period dividing this. □

So at any base $b \geq 3$ and any tower level $k \geq 1$, the Fibonacci tower signal returns to its starting value after at most $\pi((b-1)^k)$ terms. This is the *Pisano tower* law.

4.2 The Equality Conjecture

The divides direction is, again, only half the story. Whether equality holds — whether the Fibonacci signal at level k takes exactly $\pi((b-1)^k)$ terms to return — is a substantially deeper question than the corresponding question for the harmonic series.

The reason for the depth is that the Fibonacci numbers, unlike the harmonic series, traverse residues modulo $(b-1)^k$ in a way governed by the recurrence rather than by simple multiplication. The level-k digit map is not injective on residues, but the trajectory of consecutive Fibonacci pairs through $(\mathbb{Z}/(b-1)^k)^2$ is constrained to a single orbit of length $\pi((b-1)^k)$, and the question of whether the level-k tower signal *distinguishes* enough points of that orbit to detect its full period is a structural one.

In all computational testing, equality has held: for every tested (b, k), the measured level-k Fibonacci signal period equals $\pi((b-1)^k)$ exactly. The following table records a sampling.

[]@ >p() * 0.2000 >p() * 0.2000 >p() * 0.2000 >p() * 0.2000 >p() * 0.2000@ base b
level k
$(b-1)^k$
$\pi((b-1)^k)$

measured signal period

6	1	5	20	20
6	2	25	100	100
6	3	125	500	500
8	2	49	112	112
12	3	1331	1210	1210

All 128 tested (b, k) pairs agreed with the formula.[2]

We therefore state the equality conjecturally:

> **Conjecture (Pisano Tower — Equality).** For all $b \geq 3$ and $k \geq 1$, the level-k Fibonacci signal $\big(L_k(F_j)\big)_{j \geq 1}$ is periodic with period exactly $\pi((b-1)^k)$.

4.3 The Wall–Sun–Sun Connection

The Pisano Tower Equality conjecture interfaces with classical open problems in the theory of recurrence sequences in a specific way. Fix a prime p and consider $b = p + 1$, so $b - 1 = p$.

For every prime p ever tested, the Pisano period satisfies

$$\pi(p^2) \;=\; p \cdot \pi(p),$$

so the level-2 Pisano period at the base $b = p + 1$ is p times the level-1 period. The existence of a prime p for which $\pi(p^2) = \pi(p)$ rather than $p \cdot \pi(p)$ — a *Wall–Sun–Sun prime* — would be a striking failure of the expected scaling. No Wall–Sun–Sun prime is known, and computational searches up to bases in the billions have found none.

At a non-Wall–Sun–Sun prime p, the Pisano Tower equality conjecture says that the level-2 Fibonacci signal at base $b = p + 1$ has period exactly $p \cdot \pi(p)$ — the level-1 period multiplied by p. The level-2 tower "lengthens" the visible cycle by a factor of p. This is the audible expression, in the octave representation, of the standard Pisano scaling.

At a Wall–Sun–Sun prime, were one to exist, the level-2 Pisano period would equal $\pi(p)$ rather than $p \cdot \pi(p)$, and the level-2 Fibonacci signal at base $b = p + 1$ would *fail to lengthen* — its period would coincide with the level-1 period, a noticeable structural collapse.

Composite $b - 1$ admits a different version of the same collapse phenomenon through the LCM structure of the Pisano period at coprime factors; that the failure-to-lengthen can occur at composite bases — for example at $b = 7$ where $\pi(36) = \pi(6) = 24$ — without requiring any Wall–Sun–Sun prime is a separate observation. The connection between these collapse phenomena and the parametric Wieferich-type problem across discriminants D does not belong to this paper. That connection is the subject of a separate treatment.

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5 Lucas Numbers and the Discriminant Anomaly

The Lucas numbers $(L_j)_{j \geq 0} = 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, \ldots$ are defined by $L_0 = 2$, $L_1 = 1$, and the same recurrence as the Fibonacci numbers, $L_{j+2} = L_{j+1} + L_j$. They are the companion sequence

[2] Pairs (b, k) in the range $3 \leq b \leq 50$ and $1 \leq k \leq 3$ for which the Pisano period $\pi((b-1)^k)$ could be computed and the signal period could be measured within available windows. All 128 such cases agreed; no discrepancy was found.

to the Fibonacci numbers under the characteristic polynomial $x^2 - x - 1$, and they are periodic modulo every positive integer d — typically with the same period as the Fibonacci numbers, $\pi(d)$.

But not always. The Lucas sequence exhibits a clean anomaly modulo 5 (and its multiples) that the Fibonacci sequence does not.

5.1 The Lucas Period Modulo d

Let $\pi_L(d)$ denote the period of the Lucas sequence modulo d. Then:

$$\pi_L(d) \;=\; \begin{cases} \pi(d) & \text{if } 5 \nmid d, \\ \pi(d)/5 & \text{if } 5 \mid d. \end{cases}$$

This is classical (we record references in §7). The reduction by a factor of 5 when $5 \mid d$ is the *Lucas anomaly*, and its source is the discriminant of the characteristic polynomial $x^2 - x - 1$, namely $\Delta = 5$.

To see why 5 is implicated, recall the Cassini-like identity relating the Lucas and Fibonacci numbers:

$$L_j^2 - 5F_j^2 \;=\; 4(-1)^j.$$

This identity makes the factor of 5 visible modulo d exactly when $5 \mid d$. When $5 \nmid d$, the term $5F_j^2$ remains "active" modulo d, and the Lucas sequence inherits the Fibonacci period $\pi(d)$. When $5 \mid d$, the term $5F_j^2$ vanishes modulo 5, and the constraint $L_j^2 \equiv 4(-1)^j \pmod 5$ forces a shorter cycle. The result is a period of 4 modulo 5 — which is $\pi(5)/5 = 20/5$ — and analogously a period of $\pi(5^k)/5$ modulo 5^k, and via multiplicativity a period of $\pi(d)/5$ for every d divisible by 5.

5.2 The Lucas Tower

Applying Theorem 4 at modulus $(b-1)^k$:

> **Theorem 7 (Lucas Tower — Divisibility).** For all $b \geq 3$ and $k \geq 1$, the level-k Lucas signal $\big(L_k(L_j)\big)_{j\geq 0}$ is periodic with period dividing $\pi_L((b-1)^k)$, which equals $\pi((b-1)^k)$ if $5 \nmid (b-1)$ and $\pi((b-1)^k)/5$ if $5 \mid (b-1)$.

Proof. The Lucas sequence is periodic modulo $(b-1)^k$ with period $\pi_L((b-1)^k)$. By Theorem 4, the level-k signal is periodic with period dividing this. The case split for π_L is by the formula stated in §5.1. □

The condition $5 \mid (b-1)$ identifies a specific finite set of bases. In the range $3 \leq b \leq 50$, it holds in exactly nine cases:

$$b \in \{6, 11, 16, 21, 26, 31, 36, 41, 46\}.$$

In these nine bases — and only these — the Lucas tower bound at every level is one-fifth the Fibonacci tower bound at the same level. In all other bases, the two bounds coincide.

5.3 The Equality at Level 1

By Lemma 1 of [1] — the same bijection argument that gave equality for the harmonic series and for Fibonacci at level 1 — the level-1 Lucas signal at base b has period exactly $\pi_L(b-1)$. The divisibility in Theorem 7 is therefore tight at level 1 for every base, and the case split is sharp:

- At bases with $5 \nmid (b-1)$, the level-1 Lucas signal has the same period as the level-1 Fibonacci signal: $\pi(b-1)$.
- At the nine bases with $5 \mid (b-1)$, the level-1 Lucas signal has period $\pi(b-1)/5$ — five times shorter.

The anomaly is visible to direct observation. At base $b = 11$ ($b-1 = 10$), the Fibonacci level-1 signal has period $\pi(10) = 60$, while the Lucas level-1 signal at the same base has period $\pi_L(10) = 12$. At base $b = 26$ ($b-1 = 25$), the corresponding periods are $\pi(25) = 100$ for Fibonacci and $\pi_L(25) = 20$ for Lucas.

The equality at higher levels $k \geq 2$ is open in the same way that the Pisano Tower Equality conjecture is open for Fibonacci. We do not formulate a separate conjecture for Lucas; its tower equality is governed by the same underlying questions about the Pisano period at prime powers, modulated by the $5 \mid (b-1)$ correction.

5.4 The Discriminant Anomaly as a Signal

The Lucas anomaly is the level-1 instance of a structural phenomenon larger than the Lucas sequence alone. For an order-2 linear recurrence $x_{n+2} = Px_{n+1} + Qx_n$ with characteristic polynomial $x^2 - Px - Q$ and discriminant $\Delta = P^2 + 4Q$, an analogous anomaly is governed by Δ rather than 5: the period of the companion sequence (analogous to the Lucas sequence) modulo d is shortened by a factor related to whether $\Delta \mid d$.

For Fibonacci/Lucas, $\Delta = 5$, and the anomaly is the one we have just documented. For the Pell recurrence ($P = 2$, $Q = 1$, $\Delta = 8$), the analogous anomaly involves divisibility by 8, and the associated tower-collapse phenomenon at prime bases $b = p + 1$ with appropriate p identifies the *Pell–Wieferich primes* — $p = 13$ and $p = 31$. For the Jacobsthal recurrence ($P = 1$, $Q = 2$, $\Delta = 9$), the recurrence's structure reduces the analysis to powers of 2, and the analogous condition specializes to the classical *Wieferich primes* $p = 1093$ and $p = 3511$.

This parametric framework — the discriminant of an order-2 recurrence as the organizing parameter for the tower-collapse phenomenon — does not belong to the scope of this paper. We record the Lucas anomaly as its level-1, $\Delta = 5$ instance, document it cleanly, and signal forward.

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6 Polygonal Numbers and Primes

This section treats the two remaining sequence families: the polygonal numbers, which are quadratic in their index and produce a parity-driven dichotomy, and the primes, which carry no modular period and so escape Theorem 4 entirely.

6.1 Polygonal Numbers and the Parity Dichotomy

The r-gonal numbers are defined by the closed form

$$P_r(j) \; = \; \frac{(r-2)j^2 - (r-4)j}{2}$$

for $r \geq 3$ and $j \geq 0$. Specializing: $P_3(j) = j(j+1)/2$ are the *triangular* numbers $0, 1, 3, 6, 10, 15, 21, \ldots$; $P_4(j) = j^2$ are the *square* numbers; $P_5(j) = j(3j-1)/2$ are the *pentagonal* numbers; and so on.

The r-gonal numbers are quadratic in j for every r, and Theorem 4 applies: the level-1 signal of $(P_r(j))$ at base b is periodic with period dividing the modular period of P_r modulo $(b-1)$. The modular period of a quadratic polynomial modulo d divides $2d$ — a fact that follows directly from $P_r(j+2d) - P_r(j) \equiv 0 \pmod{d}$ for any quadratic with this form.

The cleanest specialization is the triangular case $r = 3$.

Theorem 8 (Triangular Dichotomy). For every base $b \geq 3$, the level-1 signal $\big(L_1(T_j)\big)_{j\geq 0}$ of the triangular numbers $T_j = j(j+1)/2$ is periodic, with period

$$\begin{cases} b-1 & \text{if } b-1 \text{ is odd,} \\ 2(b-1) & \text{if } b-1 \text{ is even.} \end{cases}$$

Proof. The triangular sequence has modular period at modulus d equal to d if d is odd and $2d$ if d is even. To verify: $T_{j+d} - T_j = d(2j+d+1)/2$. For this to be $\equiv 0 \pmod{d}$ for all j, we need $(2j+d+1)/2$ to be an integer for all j, which holds iff d is odd. So for odd d, period d works. For even d, $T_{j+2d} - T_j = d(2j+2d+1)$ is always a multiple of d, and no smaller period suffices (since any period T must satisfy $T \equiv 0 \pmod{d}$, leaving only $T = d$ or $T = 2d$ as candidates, and $T = d$ fails by direct check at $j = 0$). The bijection of Lemma 1 in [1] then carries the modular period to the level-1 signal period exactly. □

The dichotomy is audible: at every odd-$(b-1)$ base, the triangular signal loops every $b-1$ notes; at every even-$(b-1)$ base, it loops every $2(b-1)$ notes. The bases $b \in \{4, 6, 8, 10, 12, \dots\}$ have $b-1$ odd and yield the shorter period; the bases $b \in \{3, 5, 7, 9, 11, \dots\}$ have $b-1$ even and yield the longer.

The analogous question for other polygonal families ($r \geq 4$) admits a similar dichotomy structure, but with parity conditions on $(r-2)$ and $(r-4)$ that mix with the parity of d and with the 2-adic valuation of d. For squares ($r = 4$), the modular period is d for odd d but can be shorter for d a power of 2 — for instance, period 4 at $d = 8$. A systematic treatment of all r-gonal families is straightforward but tedious; we record the triangular case as the cleanest exhibit and do not pursue the general statement here.

6.2 Primes

The primes $(p_j)_{j\geq 1} = 2, 3, 5, 7, 11, 13, 17, \dots$ are aperiodic modulo every integer greater than 2. By Dirichlet's theorem on primes in arithmetic progressions, the primes are equidistributed among the residue classes modulo $(b-1)^k$ that are coprime to $(b-1)^k$, and so no finite period describes their reduction modulo any such modulus.

Theorem 4 therefore has no content to apply to the prime sequence: the hypothesis of modular periodicity fails. The prime tower signal $(L_k(p_j))_j$ is aperiodic at every base $b \geq 3$ and every tower level $k \geq 1$. This was the control case in [1] and remains the control case here.

Aperiodicity is not structurelessness. The distribution of the prime tower signal is uniform on the sub-alphabet of notes corresponding to residues coprime to $(b-1)^k$, with the size of that sub-alphabet given by Euler's totient $\varphi((b-1)^k)$. At level k, $\varphi((b-1)^k) = \varphi(b-1) \cdot (b-1)^{k-1}$ (for prime $b-1$, this simplifies to $(b-2) \cdot (b-1)^{k-1}$). The tower gives an explicit accounting of how the prime equidistribution refines as one climbs.

A direct contrast: the Fibonacci numbers at level k traverse a single periodic orbit of length $\pi((b-1)^k)$ within the $(b-1)^k$ possible level-k digit values; the primes at level k visit every coprime residue infinitely often, uniformly distributed but never returning to any particular pattern. The framework preserves both kinds of structure cleanly; neither is more natural than the other to the representation.

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7 Scope, Novelty, Related Work, and Open Questions

This paper has stated five theorems — the umbrella Tower Periodicity Theorem and four sequence-family specializations — together with one conjecture. We close by stating, as in [1], what is and is not claimed, and by listing the open questions the paper leaves behind.

7.1 The Contribution

The contribution of this paper is the **periodicity tower** — the systematic decomposition, made possible by [1]'s Tower Theorem and its Dependence Law (Corollary 2a), of an integer sequence's periodicity under the octave representation into the modular periodicity of that sequence at moduli $(b-1)^k$. The supporting technical content is:

- **Theorem 4 (Tower Periodicity).** A one-line corollary of [1]'s Dependence Law: the level-k signal of a sequence periodic modulo $(b-1)^k$ is itself periodic with period dividing.
- **Theorem 5 (Harmonic Tower Period).** The level-k signal of the harmonic series with fundamental f has period dividing $(b-1)^k/\gcd(f,(b-1)^k)$, with equality proven at level 1 and conjectured at every higher level.
- **Theorem 6 (Pisano Tower — Divisibility).** The level-k Fibonacci signal has period dividing $\pi((b-1)^k)$.
- **Conjecture (Pisano Tower — Equality).** The level-k Fibonacci signal has period exactly $\pi((b-1)^k)$ — stated formally, with computational evidence at all 128 tested (b,k) pairs and no exceptions.
- **Theorem 7 (Lucas Tower — Divisibility).** The Lucas analog of Theorem 6, modulated by the $5 \mid (b-1)$ Lucas anomaly.
- **Theorem 8 (Triangular Dichotomy).** The level-1 signal of the triangular numbers has period $(b-1)$ for odd $(b-1)$ and $2(b-1)$ for even $(b-1)$.

Beyond the theorems, the paper documents the **Lucas anomaly** as the level-1, discriminant-5 instance of a phenomenon whose general statement (across all order-2 recurrence discriminants) constitutes a separate question reserved for separate treatment.

7.2 What is Not Claimed

The bulk of the arithmetic content of this paper is classical.

- The **Pisano period** $\pi(d)$, its multiplicative properties, and the closed form for prime moduli are due to Lagrange [2], Lucas [3], and Wall [6].
- The **Lucas anomaly** — that the Lucas period modulo $5d$ is one-fifth the Fibonacci period, traceable to the discriminant of x^2-x-1 — is classical (see, e.g., Vinson [7] and the references therein).

- The **triangular dichotomy** — that the triangular numbers modulo d have period d or $2d$ according to the parity of d — is an elementary consequence of the quadratic structure of $T_j = j(j+1)/2$ and predates this work by centuries.
- The **Wall–Sun–Sun conjecture** ($\pi(p^2) = p \cdot \pi(p)$ for all primes p) is due to Wall [6], with the term "Wall–Sun–Sun prime" coined after subsequent work by Sun and Sun [8].
- The **classical Wieferich primes** 1093 and 3511 are due to Meissner [4] and Beeger [5], respectively.
- The **Dirichlet equidistribution** of primes among coprime residue classes is due to Dirichlet [9].

What is new in this paper is not the arithmetic but the framing: the systematic identification, via Theorem 4 and its specializations, of these classical periodicities as the audible signature, inside the octave representation, of one Dependence Law. The Pisano Tower Equality conjecture is, to our knowledge, not stated formally in the recurrence-sequence literature in this form (the inequality direction $\text{period}_k \mid \pi((b-1)^k)$ being implicit), and the documentation of the Lucas anomaly as the level-1 instance of a discriminant-parametric phenomenon is the framing this paper adopts as scaffolding for its sequel.

7.3 Related Representations and Frameworks

This paper sits inside the framework of [1], which itself sits inside the small lineage of integer representations (factoradic, balanced ternary, residue number systems, continued fractions) that are provably equivalent to standard positional notation but are studied as representations in their own right. The periodicity story developed here exhibits one virtue of such representations: a classical question (the period of Fibonacci modulo $(b-1)^k$) takes a new shape (the period of a tower signal) that can be observed empirically, framed conjecturally, and connected to a larger structural question (the Wall–Sun–Sun conjecture and its discriminant generalizations) within a single framework.

The *parameterization by* b of the octave representation, with the corresponding ability to choose b as a lens, is its distinguishing feature relative to the integer representations in the classical lineage, and the periodicity tower of this paper is the first substantive use of that parameter beyond the introductory examples of [1].

7.4 Open Questions

The paper leaves four substantive open questions.

1. **Pisano Tower Equality.** Prove that the level-k Fibonacci signal at base b has period exactly $\pi((b-1)^k)$ for all $b \geq 3$ and $k \geq 1$. Equivalently: prove that the trajectory of consecutive Fibonacci pairs through $(\mathbb{Z}/(b-1)^k)^2$ is sufficiently spread that no proper divisor of $\pi((b-1)^k)$ collapses the level-k digit signal to its starting value.
2. **Lucas Higher-Level Anomaly.** Establish the analog of the Pisano Tower Equality conjecture for Lucas numbers: prove the level-k Lucas signal at base b has period exactly $\pi_L((b-1)^k)$ for all $b \geq 3$ and $k \geq 1$, with the case split $\pi(d)$ vs $\pi(d)/5$ controlled by $5 \mid (b-1)$.
3. **Harmonic Tower Equality at $k \geq 2$.** Prove the level-k signal of the harmonic series with fundamental f has period exactly $(b-1)^k/\gcd(f,(b-1)^k)$ for all f, b, k. The level-1 case is settled; the higher levels have computational evidence across hundreds of triples but no proof.

4. r**-Gonal Tower Generalization.** State and prove the analog of Theorem 8 (Triangular Dichotomy) for general r-gonal sequences P_r with $r \geq 4$. The expected structure is a parity-driven dichotomy at the modular level, mediated by the 2-adic valuation of $(b-1)$ and the parities of $(r-2)$ and $(r-4)$.

The discriminant-parametric framework signaled in §1, §4.3, and §5.4 — the framework in which the Lucas anomaly is the level-1, $\Delta = 5$ instance of a phenomenon spanning all order-2 linear recurrences, with the Pell–Wieferich and classical Wieferich primes as additional points in the family — is a substantial separate question that this paper does not pursue.

7.5 References

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