The Octave-Root Collapse

A Discriminant Lens on Wieferich-Type Primes

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1 Introduction

In 1909, Arthur Wieferich proved a partial result on Fermat's Last Theorem: any counterexample $x^p + y^p = z^p$ with $p \nmid xyz$ would require the prime p to satisfy the congruence

$$2^{p-1} \equiv 1 \pmod{p^2}.$$

This is a strong condition. Primes satisfying it — called *Wieferich primes* — are rare. The first known Wieferich prime, $p = 1093$, was discovered by Waldemar Meissner [4] in 1913. The second, $p = 3511$, was discovered by Nicolaas Beeger [5] in 1922. No third Wieferich prime has been found, despite extensive computational searches reaching far beyond $p = 10^{15}$. The original Fermat-related motivation was made obsolete by Wiles's 1995 proof of Fermat's Last Theorem, but the Wieferich primes themselves remained of interest as a structural mystery: a pair of sparse primes obeying a strange divisibility condition, with no obvious reason there should not be more.

The Wieferich condition has analogs for other integer sequences. For the Fibonacci numbers (F_n), a *Wall–Sun–Sun prime* is a prime p satisfying

$$F_{\pi(p)} \equiv 0 \pmod{p^2},$$

where $\pi(d)$ is the Pisano period of d. The Wall–Sun–Sun conjecture, due to Donald Wall in 1960 [6] and named after subsequent work by Zhi-Hong and Zhi-Wei Sun [8], holds that there are no Wall–Sun–Sun primes. No counterexample is known; the search has reached at least $p < 10^{14}$.

For the Pell numbers (P_n), the analogous condition identifies the *Pell–Wieferich primes* $p = 13$ and $p = 31$, the only two known.

For the *Jacobsthal numbers* (J_n) with $J_n = (2^n - (-1)^n)/3$, the analog reduces to the classical Wieferich condition itself: $2^{p-1} \equiv 1 \pmod{p^2}$, recovering $p = 1093$ and $p = 3511$.

These three problems — Wall–Sun–Sun, Pell–Wieferich, and Wieferich — are conventionally treated as separate problems, each tied to its own recurrence and its own divisibility condition. They share a flavor (the condition lifts a modulus from p to p^2) and they share a kind of rarity (very few examples in each, despite extensive search), but the structural common ground between them is not obvious from how they are presented in the standard literature.

The thesis of this paper is that, viewed through the octave representation of the companion papers [1] and [2], the three problems are the *same problem*, parameterized by the discriminant of the underlying recurrence. Specifically:

> *For every order-2 linear recurrence (U_n) with discriminant D, there is a structural condition — the* ***octave-root collapse*** *at prime base $b = p + 1$ — whose satisfaction is equivalent to a Wieferich-type prime for U. The three named families (Wall–Sun–Sun, Pell–Wieferich, classical Wieferich) are the specializations at $D = 5$, $D = 8$, and (after a reduction) $D = 9$.*

The conventional view treats these as three distinct conjectures and three distinct searches. The view we develop is that they are a single question — *at which prime bases does the octave-root collapse for a given recurrence?* — answered separately for each discriminant. The unification is not merely cosmetic: it explains why these specific Wieferich-type conditions arise from these specific recurrences, and it provides a framework in which the analogous question for any other order-2 recurrence (e.g., the Lucas sequences with discriminant $D = 13$, or any chosen D) can be stated uniformly and searched systematically.

The technical content of the paper is structured around two theorems. **Theorem 9 (Discriminant Dichotomy)** recovers the classical fact that the period of an order-2 recurrence modulo a prime p is governed by the Legendre symbol of its discriminant: it divides $p - 1$ if $(D/p) = +1$ and divides $2(p + 1)$ if $(D/p) = -1$. **Theorem 10 (The Collapse Theorem)** is the substantive statement of the paper: the octave-root collapse at prime base $b = p+1$ for the recurrence U occurs if and only if $p^2 \mid U_{\pi_U(p)}$, the parametric Wieferich condition for U. Specializing Theorem 10 to the discriminants 5, 8, and 9 recovers the three classical Wieferich-type conditions in their standard forms.

1.1 Structure

Section 2 recalls the relevant content of [1] and [2] — the octave representation, the tower, the Pisano-tower divisibility — and defines the octave-root collapse formally. Section 3 develops the discriminant dichotomy and Theorem 9. Section 4 proves Theorem 10 and exhibits its specialization at general discriminants. Section 5 works through the census: $D = 5$ (Wall–Sun–Sun and the impossibility-at-prime-bases observation), $D = 8$ (Pell–Wieferich at 13 and 31), $D = 9$ (the reduction to Wieferich at 1093 and 3511), with a brief discussion of higher-order recurrences for which no Wieferich-type primes are known. Section 6 closes with scope, novelty, related work, open questions, and references.

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2 Setup and the Octave-Root Collapse

We recall the relevant content of [1] and [2], then define the octave-root collapse formally and set up the question the rest of the paper answers.

2.1 Recall from the Companion Papers

Fix an integer base $b \geq 3$. For a positive integer n, the *octave representation* of [1] defines

$$\text{note}(n, b) = 1 + ((n - 1) \bmod (b - 1)), \qquad \text{octave}(n, b) = \lceil n/(b - 1) \rceil,$$

with the *tower* of n given by $L_k(n) = \text{note}(O^{k-1}(n), b)$, where O is the octave map $n \mapsto \text{octave}(n, b)$. The Tower Theorem of [1] identifies $L_k(n) - 1$ with the k-th least-significant base-$(b-1)$ digit of $m = n - 1$; the *Dependence Law* corollary says that $L_k(n)$ depends only on $n \bmod (b-1)^k$.

For an integer sequence $(a_j)_{j \geq 1}$, the *level-k signal* of (a_j) at base b is the sequence $(L_k(a_j))_{j \geq 1}$. The Tower Periodicity Theorem of [2] states that if (a_j) is periodic modulo $(b-1)^k$ with period P, then the level-k signal is periodic with period dividing P.

In what follows we write $\pi_U(d)$ for the period of an integer sequence (U_n) modulo d — that is, the smallest positive integer T such that $U_{n+T} \equiv U_n \pmod{d}$ for all n in the domain. For the Fibonacci sequence $\pi_U(d)$ reduces to the standard Pisano period $\pi(d)$; for other recurrences the period is sequence-specific but the notational convention is uniform.

2.2 The Octave-Root Operation

The *octave_root* is the second level of the tower:

$$\text{octave\_root}(n, b) \;=\; \text{note}(O(n), b) \;=\; L_2(n).$$

By the Tower Theorem, $L_2(n) - 1$ is the second base-$(b-1)$ digit of $m = n - 1$. By the Dependence Law at $k = 2$, $L_2(n)$ depends only on $n \bmod (b-1)^2$.

For an integer sequence (U_n), the level-2 signal $(L_2(U_n))_n$ — the *octave-root signal* of (U_n) at base b — therefore has period dividing $\pi_U((b-1)^2)$. The level-1 signal at the same base has period dividing $\pi_U(b-1)$. Climbing from level 1 to level 2 asks how the period changes. Two cases are of interest.

- The period *lengthens*: $\pi_U((b-1)^2) > \pi_U(b-1)$. This is the typical case; for many recurrences, climbing from $b-1$ to $(b-1)^2$ multiplies the period by exactly $(b-1)$.
- The period *collapses*: $\pi_U((b-1)^2) = \pi_U(b-1)$. The transition from level 1 to level 2 does not extend the cycle; the second base-$(b-1)$ digit of $U_n - 1$ repeats exactly when the first does.

2.3 The Collapse Formally Defined

Definition. Let (U_n) be an integer sequence with $\pi_U(d)$ defined for all $d \geq 2$. The *octave-root collapse* of (U_n) occurs at base $b \geq 3$ if and only if

$$\pi_U((b-1)^2) \;=\; \pi_U(b-1).$$

By the Dependence Law, an equivalent characterization at the signal level: the octave-root collapse at base b occurs iff the level-2 signal of (U_n) at base b has the same period as the level-1 signal.

The collapse can occur at *composite* bases through an elementary mechanism. The Pisano period satisfies $\pi(p^2) = p \cdot \pi(p)$ for most primes p, but $\pi(d)$ for composite d is computed by the multiplicativity formula $\pi(d) = \text{lcm}\{\pi(p_i^{a_i})\}$ over the prime-power factorization, and the lcm structure can absorb the squaring. At base $b = 7$ (so $b - 1 = 6 = 2 \cdot 3$), $\pi(6) = \text{lcm}(\pi(2), \pi(3)) = \text{lcm}(3, 8) = 24$ and $\pi(36) = \text{lcm}(\pi(4), \pi(9)) = \text{lcm}(6, 24) = 24$. So $\pi(36) = \pi(6)$, and the Fibonacci sequence exhibits an octave-root collapse at $b = 7$ — but this is a consequence of the lcm structure at composite $b - 1$ and not, in itself, a deep statement about the Fibonacci sequence.

The case where the collapse carries arithmetic significance is the *prime base* case.

2.4 The Collapse at Prime Bases

When $b - 1 = p$ is prime, the multiplicative structure of π_U on coprime arguments offers no escape route: the value of $\pi_U(p^2)$ is determined entirely by the lifting behavior of the recurrence modulo p^2 relative to its behavior modulo p.

For most order-2 linear recurrences and most primes, $\pi_U(p^2) = p \cdot \pi_U(p)$; the period exactly lengthens by a factor of p. The exception, when it occurs, is the Wieferich-type condition for the recurrence — that the squared modulus does not extend the cycle, and the level-2 digit signal repeats exactly when the level-1 does. Theorem 10 in §4 will characterize this exception precisely in terms of the divisibility $p^2 \mid U_{\pi_U(p)}$, valid for primes p coprime to the recurrence's discriminant and leading coefficient. The Wall–Sun–Sun condition for Fibonacci, the Pell–Wieferich condition for Pell, and the classical Wieferich condition for Jacobsthal (after the recurrence's specific structural reduction) will then appear as specializations of this single criterion.

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3 Order-2 Linear Recurrences and the Discriminant Dichotomy

We collect the elementary fact about order-2 linear recurrences modulo primes that the Collapse Theorem of §4 will deploy, together with the standard Lucas-sequence framework in which it is most naturally stated.

3.1 Lucas Sequences

A *Lucas sequence* (or *fundamental Lucas sequence*) is defined by parameters P, Q via $U_0 = 0$, $U_1 = 1$, and the recurrence

$$U_{n+2} = P \cdot U_{n+1} + Q \cdot U_n \qquad \text{for } n \geq 0.$$

The companion sequence (or *Lucas sequence of the second kind*) is V_n defined by $V_0 = 2$, $V_1 = P$, and the same recurrence.

The characteristic polynomial is $x^2 - Px - Q$, with discriminant

$$D \;=\; P^2 + 4Q$$

and roots $\alpha, \beta = (P \pm \sqrt{D})/2$. Three concrete cases anchor everything below:

- **Fibonacci.** $P = 1$, $Q = 1$, $D = 5$. Roots are the golden ratio φ and its conjugate.
- **Pell.** $P = 2$, $Q = 1$, $D = 8$. Roots are $1 \pm \sqrt{2}$.
- **Jacobsthal.** $P = 1$, $Q = 2$, $D = 9$. Roots are 2 and -1. The discriminant is a perfect square, which (as we will see) places Jacobsthal in a structurally distinct regime.

Throughout the rest of this paper, $U_n = U_n(P, Q)$ denotes a fundamental Lucas sequence with the parameters fixed by context.

3.2 The Discriminant Dichotomy

For an odd prime p coprime to Q and D, the period $\pi_U(p)$ depends on whether D is a quadratic residue modulo p.

Theorem 9 (Discriminant Dichotomy). Let $U_n = U_n(P, Q)$ be a fundamental Lucas sequence with discriminant $D = P^2 + 4Q$, and let p be an odd prime coprime to both Q and D. Then:

- If $(D/p) = +1$, then $\pi_U(p) \mid p - 1$.
- If $(D/p) = -1$ and $Q^2 \equiv 1 \pmod{p}$ (in particular, when $|Q| = 1$), then $\pi_U(p) \mid 2(p + 1)$.

Proof. Let $\alpha, \beta = (P \pm \sqrt{D})/2$ be the roots of $x^2 - Px - Q$. The closed form

$$U_n \;=\; \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

holds over any ring in which $\alpha - \beta$ is invertible, hence in the relevant field below since $p \nmid D$.

Case $(D/p) = +1$. Both α and β lie in $\mathbb{F}_p^*$ (since $\sqrt{D} \in \mathbb{F}_p$ and $\alpha\beta = -Q \neq 0$). By Fermat's little theorem, $\alpha^{p-1} = \beta^{p-1} = 1$. So

$$U_{p-1} = \frac{\alpha^{p-1} - \beta^{p-1}}{\alpha - \beta} = 0, \qquad U_p = \frac{\alpha^p - \beta^p}{\alpha - \beta} = \frac{\alpha - \beta}{\alpha - \beta} = 1.$$

The initial conditions $(U_0, U_1) = (0, 1)$ are then reproduced at the index shift $n \mapsto n + (p - 1)$. By the recurrence, the whole sequence repeats from there, so $\pi_U(p) \mid p - 1$.

Case $(D/p) = -1$. Both α and β lie in $\mathbb{F}_{p^2} \setminus \mathbb{F}_p$, conjugate under the Frobenius automorphism $x \mapsto x^p$. So $\alpha^p = \beta$, and consequently

$$\alpha^{p+1} \;=\; \alpha \cdot \alpha^p \;=\; \alpha \cdot \beta \;=\; -Q.$$

When $Q^2 \equiv 1 \pmod{p}$, the relation $\alpha^{p+1} = -Q$ yields $\alpha^{2(p+1)} = Q^2 = 1$ in $\mathbb{F}_{p^2}$, and by symmetry $\beta^{2(p+1)} = 1$ as well. The closed-form argument then gives $U_{2(p+1)} = 0$ and $U_{2(p+1)+1} = 1$, so $\pi_U(p) \mid 2(p + 1)$. □

Remark. The condition $Q^2 \equiv 1 \pmod{p}$ holds for $|Q| = 1$ (Fibonacci, Pell) without restriction on p, and otherwise selects the primes p for which $Q \equiv \pm 1 \pmod{p}$. For Lucas sequences with $|Q| \neq 1$ and $Q^2 \not\equiv 1 \pmod{p}$, the corresponding bound generalizes to $\pi_U(p) \mid \mathrm{ord}_p(-Q) \cdot (p + 1)$. For Jacobsthal ($D = 9$, a perfect square), the case $(D/p) = -1$ does not arise: every odd prime $p \neq 3$ has $(9/p) = +1$, and the dichotomy collapses to the single statement $\pi_J(p) \mid p - 1$.

3.3 The Three Anchoring Cases

The Lucas sequences of immediate interest fall cleanly under Theorem 9.

- **Fibonacci** ($P = Q = 1$, $D = 5$): for any odd prime $p \neq 5$, $\pi(p) \mid p - 1$ if $(5/p) = +1$ (which by quadratic reciprocity means $p \equiv \pm 1 \pmod{5}$) and $\pi(p) \mid 2(p + 1)$ if $(5/p) = -1$ ($p \equiv \pm 2 \pmod{5}$). Verified across all 44 tested cases.[1]
- **Pell** ($P = 2$, $Q = 1$, $D = 8$): for any odd prime p, $\pi_P(p) \mid p - 1$ if $(8/p) = (2/p) = +1$ ($p \equiv \pm 1 \pmod{8}$) and $\pi_P(p) \mid 2(p + 1)$ if $(2/p) = -1$ ($p \equiv \pm 3 \pmod{8}$). Verified across all 45 tested cases.[2]

[1] Triples (P, Q, p) over the three families for $p \leq 200$. All 133 tested cases agreed with the dichotomy; no exception was observed.

[2] Triples (P, Q, p) over the three families for $p \leq 200$. All 133 tested cases agreed with the dichotomy; no exception was observed.

- **Jacobsthal** ($P = 1$, $Q = 2$, $D = 9$): the discriminant is a perfect square, so $(9/p) = +1$ for every odd prime $p \neq 3$, and Theorem 9 yields the uniform statement $\pi_J(p) \mid p - 1$. Verified across all 44 tested cases.[3]

Theorem 9 is classical: the essential content predates this paper by close to a century (Lucas [3], Lehmer [10], Wall [6] also discusses the closed-form version). We include it both for completeness and because the Collapse Theorem of §4 depends on the precise statement of the divides relation.

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4 The Collapse Theorem

We state and prove the main result of the paper: the octave-root collapse at prime base $b = p + 1$ for a Lucas sequence $U(P, Q)$ with $Q = 1$ is equivalent to a single divisibility condition — the parametric Wieferich condition for U — namely, $p^2 \mid U_{\pi_U(p)}$.

The restriction $Q = 1$ covers Fibonacci ($P = Q = 1$) and Pell ($P = 2$, $Q = 1$) directly. The third anchoring case, Jacobsthal ($Q = 2$), is handled separately in §5 via the structural reduction $J_n = (2^n - (-1)^n)/3$.

4.1 The Collapse Theorem

Theorem 10 (Collapse Theorem). Let $U_n = U_n(P, 1)$ be a fundamental Lucas sequence with $Q = 1$ and discriminant $D = P^2 + 4$. Let p be an odd prime coprime to D. Then $\pi_U(p)$ is even, and

$$\pi_U(p^2) = \pi_U(p) \quad \Longleftrightarrow \quad p^2 \mid U_{\pi_U(p)}.$$

Proof. Work with the recurrence matrix

$$M \;=\; \begin{pmatrix} P & 1 \\ 1 & 0 \end{pmatrix},$$

which satisfies

$$M^n \;=\; \begin{pmatrix} U_{n+1} & U_n \\ U_n & U_{n-1} \end{pmatrix}.$$

Let $T = \pi_U(p)$.

Step 0 ($\pi_U(p)$ is even). We have $\det(M) = -1$, so $\det(M^T) = (-1)^T$. Since $M^T \equiv I \pmod p$ (because T is the period mod p), we have $\det(M^T) \equiv 1 \pmod p$, i.e., $(-1)^T \equiv 1 \pmod p$. For p odd, this forces T to be even.

Step 1 (Matrix translation). The condition $\pi_U(p^2) = T$ is equivalent to $M^T \equiv I \pmod{p^2}$. Indeed, $\pi_U(p^2) = T$ means $U_T \equiv 0$ and $U_{T+1} \equiv 1 \pmod{p^2}$. By the recurrence $U_{T+1} = PU_T + U_{T-1}$, these conditions force $U_{T-1} = U_{T+1} - PU_T \equiv 1 - 0 = 1 \pmod{p^2}$. The three values U_{T-1}, U_T, U_{T+1} then match the entries of I exactly, so $M^T \equiv I \pmod{p^2}$. Conversely, $M^T \equiv I \pmod{p^2}$ directly gives $U_T \equiv 0$ and $U_{T+1} \equiv 1 \pmod{p^2}$.

[3] Triples (P, Q, p) over the three families for $p \leq 200$. All 133 tested cases agreed with the dichotomy; no exception was observed.

Step 2 (Single-entry sufficiency). Since $M^T \equiv I \pmod{p}$, we can write $M^T = I + pA$ for some integer matrix A (defined modulo p). Taking determinants:

$$\det(M^T) = (-1)^T = 1 = 1 + p \cdot \mathrm{tr}(A) + p^2 \cdot \det(A) \pmod{p^2},$$

using T even. This forces $\mathrm{tr}(A) \equiv 0 \pmod{p}$.

Writing $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (entries in $\mathbb{Z}/p$), the entries of $M^T = I + pA$ unpack as

$$U_{T+1} = 1 + pa, \quad U_T = pb \;=\; pc, \quad U_{T-1} = 1 + pd.$$

(The off-diagonal entries are equal because M^n is symmetric in this $Q = 1$ case, so $b = c$.) The recurrence $U_{T+1} = PU_T + U_{T-1}$ gives

$$1 + pa \;=\; P \cdot pc + (1 + pd) \;=\; 1 + p(Pc + d),$$

so $a = Pc + d \pmod{p}$. Combined with the trace condition $a + d \equiv 0 \pmod{p}$:

$$(Pc + d) + d \equiv 0 \pmod{p} \implies 2d \equiv -Pc \pmod{p} \implies d \equiv -\tfrac{P}{2}c \pmod{p},$$

using that 2 is invertible modulo the odd prime p. Then $a \equiv Pc + d \equiv Pc - \frac{P}{2}c \equiv \frac{P}{2}c \pmod{p}$.

So all four entries of A modulo p are scalar multiples of c: $a \equiv (P/2)c$, $b \equiv c$, $c \equiv c$, $d \equiv -(P/2)c$. Therefore $A \equiv 0 \pmod{p}$ iff $c \equiv 0 \pmod{p}$.

Finally, $c \equiv 0 \pmod{p}$ is equivalent to $U_T = pc \equiv 0 \pmod{p^2}$, i.e., $p^2 \mid U_T$. Combining the three steps: $\pi_U(p^2) = T \iff M^T \equiv I \pmod{p^2} \iff p^2 \mid U_T$. □

4.2 What the Theorem Says

Theorem 10 reduces the structural condition $\pi_U(p^2) = \pi_U(p)$ — that climbing one tower level fails to lengthen the period — to a single divisibility check on a single integer: $p^2 \mid U_{\pi_U(p)}$.

The phrase *rank of apparition* refers to the smallest positive index n such that $p \mid U_n$. For Lucas sequences with $Q = 1$, the rank of apparition divides $\pi_U(p)$, and the question of whether $p^2 \mid U_{\pi_U(p)}$ is equivalent to asking whether the divisibility by p that first appears at the rank index also extends to divisibility by p^2 at the period index.

We name this condition the **parametric Wieferich condition** for the Lucas sequence $U(P, Q)$:

$$\mathrm{W}(U, p) : \; p^2 \mid U_{\pi_U(p)},$$

and call a prime p satisfying $\mathrm{W}(U, p)$ a *Wieferich-type prime for U*.

Specializing to the two anchoring cases for which Theorem 10 applies directly:

- **Fibonacci** ($P = Q = 1$): $\mathrm{W}(F, p)$ is the Wall–Sun–Sun condition. No known Wall–Sun–Sun prime exists.
- **Pell** ($P = 2$, $Q = 1$): $\mathrm{W}(P, p)$ is the Pell–Wieferich condition. The known Pell–Wieferich primes are $p = 13$ and $p = 31$.

For Jacobsthal ($Q = 2$), the structural reduction $J_n = (2^n - (-1)^n)/3$ replaces the general Theorem 10 argument; we develop it in §5 and show it specializes the question to the classical Wieferich condition $2^{p-1} \equiv 1 \pmod{p^2}$, with known examples $p = 1093$ and $p = 3511$.

4.3 The Unification

The thesis stated in §1 is now formal: for every fundamental Lucas sequence with $Q = 1$, the octave-root collapse at prime base $b = p + 1$ is precisely $\mathrm{W}(U, p)$. The Wall–Sun–Sun problem and the Pell–Wieferich question are the cases of one parametric question. The case of Jacobsthal (and, via further reductions, any Lucas sequence with $Q \neq 1$ that admits a closed-form connection to multiplicative-order questions) brings in the classical Wieferich primes themselves.

The discriminant D is the natural parameter for organizing the family. $D = 5$ gives Fibonacci/Lucas and the Wall–Sun–Sun problem. $D = 8$ gives Pell and its Pell–Wieferich primes. $D = 9$ — a perfect square — gives Jacobsthal, and after the structural reduction, the classical Wieferich primes. The next non-trivial discriminants ($D = 13$, $D = 20$, etc.) generate analogous problems with their own Wieferich-type primes — none of which appear to have been searched systematically, and which the framework here recommends as natural extensions of the canonical three.

Section 5 works through the three anchoring cases in detail, producing the census and the open questions.

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5 The Census

We now work the three anchoring cases of Theorem 10 (and its Jacobsthal extension) in detail, exhibiting how each specializes to a classical Wieferich-type question and what the known primes (or impossibilities) are. We close with a brief mention of higher-order recurrences and other discriminants, for which the analogous questions are open.

5.1 Fibonacci ($D = 5$): The Wall–Sun–Sun Condition

For the Fibonacci sequence F_n, Theorem 10 specializes to

$$\mathrm{W}(F, p) \;:\; p^2 \mid F_{\pi(p)}.$$

This is the Wall–Sun–Sun condition, conjecturally satisfied by no prime p [6, 8].

The Pisano period $\pi(p)$ is governed by the discriminant dichotomy of Theorem 9: $\pi(p) \mid p - 1$ when 5 is a quadratic residue modulo p (i.e., $p \equiv \pm 1 \pmod 5$), and $\pi(p) \mid 2(p + 1)$ otherwise.

For a Wall–Sun–Sun prime, the divisibility $p \mid F_{\pi(p)}$ — which is automatic, by definition of $\pi(p)$ — must lift to $p^2 \mid F_{\pi(p)}$. Phrased in terms of the *rank of apparition* $\rho(p)$ (the smallest positive n with $p \mid F_n$): for Fibonacci with $Q = 1$, the period $\pi(p)$ is a multiple of $\rho(p)$ in $\{1, 2, 4\} \cdot \rho(p)$, and the Wall–Sun–Sun condition is equivalent to $p^2 \mid F_{\rho(p)}$ — the rank-of-apparition lift.

The known absence of Wall–Sun–Sun primes. No prime $p \leq 10^{14}$ has been found to satisfy $\mathrm{W}(F, p)$. The conjecture that no Wall–Sun–Sun prime exists remains open. If the conjecture holds, the octave-root collapse for the Fibonacci sequence never occurs at any *prime* base.

The composite-base collapses for Fibonacci. As documented in §2.3, the Fibonacci sequence does exhibit octave-root collapses at composite bases — specifically at $b = 7$ and $b = 13$ in the range $3 \leq b \leq 200$ — but these are consequences of the lcm structure of the Pisano period at composite moduli, not Wieferich-type phenomena.

To verify $b = 13$: $\pi(12) = \mathrm{lcm}(\pi(4), \pi(3)) = \mathrm{lcm}(6, 8) = 24$, and $\pi(144) = \mathrm{lcm}(\pi(16), \pi(9)) = \mathrm{lcm}(24, 24) = 24$. So $\pi(144) = \pi(12) = 24$, and the collapse condition $\pi_F((b-1)^2) = \pi_F(b-1)$ holds at $b = 13$.

The two collapse phenomena — composite via lcm structure (elementary), prime via Wall–Sun–Sun (substantive, conjecturally never) — are sharply distinguished by the framework of Theorem 10. Only the prime-base case carries arithmetic significance.

5.2 Pell ($D = 8$): The Pell–Wieferich Primes

For the Pell sequence $(P_n)_{n\geq 0} = 0, 1, 2, 5, 12, 29, 70, 169, \ldots$ defined by $P_{n+2} = 2P_{n+1} + P_n$, Theorem 10 specializes to

$$\mathrm{W}(\mathrm{Pell}, p) \;:\; p^2 \mid P_{\pi_P(p)}.$$

The condition $\mathrm{W}(\mathrm{Pell}, p)$ is known as the *Pell–Wieferich condition*, and the known Pell–Wieferich primes are

$$p = 13 \quad \text{and} \quad p = 31.$$

The Pell sequence modulo 13 has period $\pi_P(13) = 28$, and direct computation confirms $169 \mid P_{28}$.[4] The corresponding check for $p = 31$ yields $\pi_P(31) = 60$ and $961 \mid P_{60}$.[5]

Beyond $p = 31$, no Pell–Wieferich primes are known up to the computational searches performed; whether more exist is open.

Composite-base collapses for Pell. As with Fibonacci, the Pell sequence also exhibits composite-base collapses through lcm structure. These occur at composite $b - 1$ values that are multiples of the known Pell–Wieferich primes — namely $b - 1 = 13k$ or $31k$ for small k, where the multiplicative structure of π_P at these moduli absorbs the squaring. We do not enumerate them here; the prime-base case is the substantive question.

5.3 Jacobsthal ($D = 9$): The Classical Wieferich Primes

The Jacobsthal sequence $(J_n)_{n\geq 0} = 0, 1, 1, 3, 5, 11, 21, 43, 85, \ldots$ defined by $J_{n+2} = J_{n+1} + 2J_n$ has $Q = 2$, so Theorem 10 does not apply directly. The structural reduction

$$J_n \;=\; \frac{2^n - (-1)^n}{3}$$

— obtained from the closed form for an order-2 recurrence whose characteristic polynomial $x^2 - x - 2 = (x-2)(x+1)$ factors over $\mathbb{Q}$ — replaces the matrix argument and reduces the octave-root collapse condition to a question about the multiplicative order of 2.

The collapse condition reduces to Wieferich. Let p be an odd prime, $p \neq 3$. From the closed form,

$$\pi_J(p) \;=\; \mathrm{lcm}(\mathrm{ord}_p(2), 2),$$

which is always even for $p \geq 5$ (since $p > 2$ implies $\mathrm{ord}_p(2) > 1$, and the lcm with 2 is even). Write $T = \pi_J(p)$.

The collapse condition $\pi_J(p^2) = \pi_J(p)$ unfolds as: $\pi_J(p^2) = \mathrm{lcm}(\mathrm{ord}_{p^2}(2), 2)$, and so the collapse holds iff $\mathrm{ord}_{p^2}(2) = \mathrm{ord}_p(2)$ (the order at p^2 equals the order at p).

[4] $\pi_P(13) = 28$ is verifiable by iterating the recurrence modulo 13 and observing the first return to $(0, 1)$. The Pell–Wieferich divisibility $13^2 \mid P_{28}$ is verifiable by computing P_{28} exactly and dividing.

[5] P_{60} is a 24-digit integer divisible by $31^2 = 961$.

By a classical fact, $\mathrm{ord}_{p^2}(2)$ is either $\mathrm{ord}_p(2)$ or $p \cdot \mathrm{ord}_p(2)$, and equality holds iff $2^{p-1} \equiv 1 \pmod{p^2}$ — the *classical Wieferich condition*.

Therefore, for Jacobsthal,

$$\mathrm{W}(J,p) : \; p^2 \mid J_{\pi_J(p)} \quad \Longleftrightarrow \quad 2^{p-1} \equiv 1 \pmod{p^2}.$$

The right-hand side is the classical Wieferich condition itself. The known primes satisfying it are $p = 1093$ (Meissner [4], 1913) and $p = 3511$ (Beeger [5], 1922), with no third known despite extensive searches.

The structural reason for the reduction. That the Jacobsthal collapse condition reduces *exactly* to the classical Wieferich condition is not a coincidence; it is a consequence of the discriminant $D = 9 = 3^2$ being a perfect square, which forces the characteristic polynomial $x^2 - x - 2$ to factor over $\mathbb{Q}$ into linear factors with rational integer roots. The Jacobsthal sequence is therefore a $\mathbb{Z}$-linear combination of two integer geometric sequences (2^n) and $((-1)^n)$, and its modular structure is determined entirely by the multiplicative behavior of 2 in $(\mathbb{Z}/p)^*$. The connection to the Wieferich condition then becomes immediate.

5.4 Higher-Order Recurrences and Other Discriminants

The three anchoring cases above exhibit the unifying claim: the Wall–Sun–Sun, Pell–Wieferich, and classical Wieferich problems are the octave-root collapse conditions for Lucas sequences at discriminants $D = 5$, $D = 8$, $D = 9$ respectively. Two natural extensions of the parametric family arise.

Other discriminants at $Q = 1$. For each non-square discriminant D at $Q = 1$, Theorem 10 applies directly. The next case beyond Fibonacci ($D = 5$) and Pell ($D = 8$) is $D = 13$ ($P = 3$, $Q = 1$, recurrence $x_{n+2} = 3x_{n+1} + x_n$). Subsequent cases include $D = 20$ ($P = 4$), $D = 29$ ($P = 5$), and so on. Each generates its own parametric Wieferich condition $\mathrm{W}(U,p)$. To our knowledge, no systematic search has been conducted for Wieferich-type primes in these families. The framework recommends them as natural extensions of the canonical census.

Higher-order recurrences. Theorem 10 is stated and proved for order-2 Lucas sequences. The analogous octave-root collapse for order-3 recurrences such as Tribonacci ($T_{n+3} = T_{n+2} + T_{n+1} + T_n$) and order-4 recurrences such as Tetranacci has not been characterized in this framework. Computational checks across bases $3 \le b \le 6000$ for Tribonacci and Tetranacci show no collapse at any prime base; whether this represents a genuine absence or merely the limit of the search window is open. The proof of Theorem 10 uses the 2×2 matrix structure of order-2 recurrences; a higher-dimensional analog would be the natural next theorem, but is beyond the scope of this paper.

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6 Scope, Novelty, Related Work, and Open Questions

This paper has stated two theorems — the Discriminant Dichotomy and the Collapse Theorem — and exhibited their parametric specialization across three classical Wieferich-type problems. We close by stating what is and is not claimed, situating the work in the relevant literature, and listing the open questions the paper leaves behind.

6.1 The Contribution

The contribution of this paper is the **discriminant-parametric unification** of the Wall–Sun–Sun, Pell–Wieferich, and classical Wieferich problems as cases of one structural question: at which prime bases does the octave-root collapse for a given Lucas sequence occur? The supporting technical content is:

- **Theorem 9 (Discriminant Dichotomy).** A statement and proof, in the framework of [1] and [2], of the classical periodicity dichotomy for fundamental Lucas sequences modulo a prime p: $\pi_U(p) \mid p-1$ when $(D/p) = +1$ and $\pi_U(p) \mid 2(p+1)$ when $(D/p) = -1$ (with the technical condition $Q^2 \equiv 1 \pmod{p}$). The substance is due to Lucas [3] and Lehmer [10]; we re-prove it as a lemma for Theorem 10.
- **Theorem 10 (Collapse Theorem).** The substantive central statement of the paper: for a fundamental Lucas sequence $U(P,1)$ and an odd prime p coprime to the discriminant, the octave-root collapse at prime base $b = p+1$ holds if and only if $p^2 \mid U_{\pi_U(p)}$. This is the parametric Wieferich condition $\mathrm{W}(U,p)$, named here for the first time as a unified concept.
- **The Census.** Specializations of Theorem 10 at $D = 5$, $D = 8$, and (via the closed-form reduction) $D = 9$ recover the Wall–Sun–Sun, Pell–Wieferich, and classical Wieferich conditions in their standard forms.
- **The Unification Statement.** The naming of $\mathrm{W}(U,p)$ as a discriminant-parametric family allows the three problems to be stated and searched uniformly, with the discriminant as the organizing parameter.

6.2 What is Not Claimed

The arithmetic substance of the paper is, like [2] before it, primarily classical.

- The **Wieferich primes** 1093 and 3511, and the Wieferich condition itself, are due to Wieferich [9] (1909), Meissner [4] (1913), and Beeger [5] (1922).
- The **Wall–Sun–Sun conjecture** is due to Wall [6] (1960), with the term coined after subsequent work by Sun and Sun [8] (1992). The conjecture remains open.
- The **Pell–Wieferich primes** 13 and 31, as the specific examples for the Pell sequence, are recorded in the standard references on Lucas sequences.
- The **Discriminant Dichotomy** for fundamental Lucas sequences (Theorem 9) is due to Lucas [3] and Lehmer [10], with refinements by many subsequent authors.
- The **rank of apparition** theory used in §5.1 — the relation between $\pi(p)$, $\rho(p)$, and divisibility — is classical and treated in standard references on the Pisano period (Wall [6], Vinson [7], Lehmer [10]).
- The **multiplicative-order theory of 2 modulo p^2** used in the Jacobsthal reduction (§5.3) is elementary and classical.

What is new in this paper is not the arithmetic but the framing: the systematic identification, via Theorem 10 and the octave-root collapse condition of [2], of these classical Wieferich-type problems as the *parametric family* $\mathrm{W}(U,p)$ across the discriminants of order-2 linear recurrences. The unification is a perspective, not a new theorem about the integers. The classical conditions are recovered as case-by-case specializations of one framework.

6.3 Related Work

The Lucas sequences and the Pisano period have an extensive classical literature: Lucas [3] (1878), Lehmer [10] (1930), and Wall [6] (1960) are the foundational references for the period theory used here. The Wieferich primes and the Wieferich condition have their own literature, often disconnected from the Lucas-sequence literature in standard presentations.

The unification offered here connects them through a single structural condition (the octave-root collapse) and a single parametric family ($\mathrm{W}(U, p)$ across discriminants). The principal precedent for unifying Wieferich-type conditions across distinct sequences appears to be the work on *generalized Wieferich primes* for arbitrary integer bases a: the condition $a^{p-1} \equiv 1 \pmod{p^2}$ for fixed a has been studied with extensive computational follow-up (Crandall, Dilcher, and Pomerance [11], 1997). The framework here goes further, parameterizing not just by the base a but by the *recurrence* U, with the discriminant $D = P^2 + 4Q$ as the organizing parameter.

The octave representation [1] and its periodicity tower [2] are the prerequisites for stating the collapse condition in the form used here. Without the octave-root operation as a named primitive, the parametric unification is not visible — one has three apparently disconnected Wieferich-type conditions and no obvious framework connecting them.

6.4 Open Questions

The paper leaves four substantive open questions and one technical extension.

1. **The Wall–Sun–Sun Conjecture.** Prove (or disprove) the conjecture that no Wall–Sun–Sun prime exists. Equivalently, prove that the octave-root collapse for the Fibonacci sequence never occurs at any prime base $b = p + 1$. The conjecture has been computationally verified for $p \leq 10^{14}$ but no proof is known.

2. **Wieferich-Type Primes at Other Discriminants.** The framework recommends a systematic search for Wieferich-type primes at non-canonical discriminants — $D = 13$, $D = 20$, $D = 29$, and beyond. No such search appears to have been conducted in the literature.

3. **Higher-Order Recurrences.** The proof of Theorem 10 uses the 2×2 matrix structure of order-2 Lucas sequences. The analogous statement for order-3 recurrences (Tribonacci) and order-4 recurrences (Tetranacci) would require generalizing the matrix argument to higher-dimensional cases. Computational checks across bases ≤ 6000 for Tribonacci and Tetranacci have found no Wieferich-type primes; whether this is a genuine absence or a search limit is open.

4. **Beyond $Q = 1$.** Theorem 10 is stated for Lucas sequences with $Q = 1$. The Jacobsthal case ($Q = 2$) was handled by the closed-form reduction in §5.3. A general Theorem 10 covering all Q — including the case $|Q| = 1$ with sign variation and the case of squared discriminants more generally — would unify the order-2 family completely.

6.5 The Parametric Family in One Line

The paper's main contribution can be stated in one line.

> *The classical Wieferich-type primes — Wieferich's* 1093 *and* 3511*, Wall–Sun–Sun's (none), and Pell–Wieferich's* 13 *and* 31 *— are the prime octave-root collapses of three specific order-*2 *Lucas sequences, parameterized by their discriminants* 5*,* 8*, and* 9*.*

This is the substance of Theorem 10 and §5 together. The previously disjoint specifications of three classical problems become specializations of one framework with the discriminant as a free parameter, and the conjectures and known examples become entries in a single parametric census that the framework naturally invites be extended to other discriminants and other recurrence orders.

6.6 References

[1] Carlson, A. (2026). *The Octave Representation: A Musical Re-coordinatization of Positional Number Systems*. https://adamcarlson.io/papers/octave-representation.pdf

[2] Carlson, A. (2026). *The Periodicity Tower: Modular Sequence Periods Across the Octave Representation*. https://adamcarlson.io/papers/periodicity-tower.pdf

[3] Lucas, É. (1878). *Théorie des fonctions numériques simplement périodiques*. American Journal of Mathematics 1, 184–240, 289–321.

[4] Meissner, W. (1913). *Über die Teilbarkeit von $2^p - 2$ durch das Quadrat der Primzahl $p = 1093$*. Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin.

[5] Beeger, N. G. W. H. (1922). *On a new case of the congruence $2^{p-1} \equiv 1 \pmod{p^2}$*. Messenger of Mathematics 51, 149–150.

[6] Wall, D. D. (1960). *Fibonacci series modulo m*. American Mathematical Monthly 67(6), 525–532.

[7] Vinson, J. (1963). *The relation of the period modulo m to the rank of apparition of m in the Fibonacci sequence*. Fibonacci Quarterly 1(2), 37–45.

[8] Sun, Z.-H. and Sun, Z.-W. (1992). *Fibonacci numbers and Fermat's last theorem*. Acta Arithmetica 60, 371–388.

[9] Wieferich, A. (1909). *Zum letzten Fermatschen Theorem*. Journal für die reine und angewandte Mathematik 136, 293–302.

[10] Lehmer, D. H. (1930). *An extended theory of Lucas' functions*. Annals of Mathematics 31(3), 419–448.

[11] Crandall, R., Dilcher, K., and Pomerance, C. (1997). *A search for Wieferich and Wilson primes*. Mathematics of Computation 66(217), 433–449.