Musical and Combinatorial Readings of the Octave Representation

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1 Introduction

The companion papers in this series have developed the octave representation through theorems. The first paper [1] introduced the representation and proved its equivalence to base-$(b-1)$ positional notation. The second paper [2] developed the periodicity tower and characterized how integer sequences appear inside the representation through their modular structure. The third paper [3] turned the framework on a parametric family of classical questions, identifying Wall–Sun–Sun, Pell–Wieferich, and Wieferich primes as specializations of one structural condition.

This paper changes register. The previous three were theorem-driven; this one is exhibits-driven. The aim is not to prove a new theorem about the integers but to display, on concrete sequences from across number theory and combinatorics, what the octave representation makes visible — or audible — when the right base is chosen. The structural claims we report are classical in every case; the contribution of the paper is the exhibition itself, and the demonstration that a single representation suffices to make four different kinds of arithmetic phenomena directly perceptible.

The four exhibits are:

- **Interval Cycles and Musical Modes (§2).** The harmonic series specialization of [2] takes a musical reading when the base is interpreted as a tuning system. At base $b = 13$ (so $b-1 = 12$, the chromatic system of Western tonal music), the harmonic fundamentals $f \in \{1, 2, \ldots, 12\}$ trace out the named musical modes: $f = 1$ is the chromatic scale, $f = 7$ is the circle of fifths, $f = 2$ is the whole-tone scale, $f = 5$ is the circle of fourths. The bases coprime to 12 — there are $\varphi(12) = 4$ of them — produce full-cycle modes. This generalizes to every base: each base b has $\varphi(b-1)$ full modes, and the modal structure is a direct consequence of the period-$(b-1)/\gcd(f, b-1)$ result from [2].
- **Ramanujan's Partition Congruences as Stripes (§3).** The partition function $p(n)$ counts the number of ways to write n as a sum of positive integers (order-independent). Ramanujan's celebrated 1919 congruences state that $p(5k+4) \equiv 0 \pmod 5$, $p(7k+5) \equiv 0 \pmod 7$, and $p(11k+6) \equiv 0 \pmod{11}$. In the octave representation, each congruence manifests as a "stripe" — a regular arithmetic progression of indices n at which the note signal $\mathrm{note}(p(n), b)$ takes the fixed value $b-1$ (the highest note in the alphabet, corresponding to residue 0 modulo $b-1$). At base $b = 6$, the congruence $p(5k+4) \equiv 0 \pmod 5$ guarantees a stripe of value 5 at every

index $n \equiv 4 \pmod 5$. Bases $b = 8$ and $b = 12$ produce analogous stripes for the modulus-7 and modulus-11 congruences.

- **Catalan Parity as a Powers-of-2 Indicator (§4).** The Catalan numbers C_n have a striking parity property: C_n is odd if and only if $n+1$ is a power of 2, i.e., $n \in \{0, 1, 3, 7, 15, 31, \dots\}$. In the octave representation at base $b = 3$ — the smallest non-degenerate base — the note signal $\mathrm{note}(C_n, 3)$ is simply 1 when C_n is odd and 2 when C_n is even. The signal therefore acts as a direct *powers-of-2 indicator*: every appearance of the value 1 marks an index $n = 2^k - 1$, and the structure becomes a discrete sequence of pulses spaced at the Mersenne-like positions.
- **The Oliver–Soundararajan Prime Last-Digit Bias (§5).** A surprising result of Robert Lemke Oliver and Kannan Soundararajan (2016) established that consecutive primes exhibit a slight statistical bias against sharing their last decimal digit — a deviation from the uniform distribution one might naively expect. In the octave representation at base $b = 11$ (so $b - 1 = 10$), the note of any prime $p > 5$ equals exactly the last decimal digit of p. The Oliver–Soundararajan bias is therefore directly observable in the consecutive-prime note signal at base 11, and analogous biases at other moduli become directly observable at the corresponding bases.

The exhibits are deliberately heterogeneous: a music-theory result, a partition-theory congruence, a combinatorial-number-theory parity statement, an analytic-number-theory bias. They share no common subject matter. They share only the framework — the octave representation and the choice of base b as a lens. In each case, the structural fact is classical, and the demonstration is that the right base makes it directly visible in note coordinates.

1.1 Structure

Section 2 develops the interval-cycle / musical-mode exhibit in detail, recalling the harmonic-series specialization from [2] and showing how the $\varphi(b-1)$ modes at each base specialize to named scales at base 13. Section 3 displays Ramanujan's three classical congruences as stripes in the partition note signal at bases $6, 8, 12$. Section 4 works the Catalan parity exhibit at base 3, including a brief proof of the powers-of-2 parity rule for completeness. Section 5 covers the Oliver–Soundararajan bias and its note-signal manifestation at base 11. Section 6 closes with the observation that ties the exhibits together: in each case, the *choice of base* is what makes the underlying modular fact perceptible, and different bases reveal different facets. Section 7 records scope, related work, and references.

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2 Interval Cycles and Musical Modes

The harmonic series with fundamental f produced, in the framework of [2], a level-1 note signal at base b with period $(b-1)/\gcd(f, b-1)$. This section interprets that result musically: when the base b is treated as a tuning system with $b - 1$ notes per octave, the harmonic-series note signal traces out a *musical mode*, and the choice of fundamental f selects which mode.

2.1 The Harmonic Series in Musical Coordinates

Let $b \geq 3$ be a fixed base, and let f be a positive integer fundamental. The harmonic series $(jf)_{j\geq 1} = f, 2f, 3f, \ldots$ in the octave representation produces the note signal

$$\mathrm{note}(jf, b) \;=\; 1 + ((jf - 1) \bmod (b - 1)),$$

which by Theorem 5 of [2] is periodic with period

$$T(f, b) \;=\; \frac{b-1}{\gcd(f, b-1)}.$$

We call this signal the *mode of fundamental f at base b*, and we call the cyclic sequence of $T(f, b)$ distinct note values that it visits the *mode* itself.

The musical reading interprets the base b as a tuning system in which the octave is divided into $b - 1$ equal steps (semitones). At base $b = 13$, this is the chromatic system of Western tonal music: 12 semitones per octave. At base $b = 8$, the system has 7 steps per octave, which approximates the diatonic system; at base $b = 6$, the system has 5 steps per octave, the pentatonic system; at base $b = 20$, the system has 19 steps per octave, recovering nineteen-tone equal temperament. The harmonic series in note coordinates corresponds, in this reading, to ascending the system by fixed intervals of f semitones at each step.

2.2 The Twelve Modes at Base 13

At base $b = 13$, with $b-1 = 12$ semitones per octave, the harmonic-series note signal for fundamental f has period $T(f, 13) = 12/\gcd(f, 12)$. The twelve possible fundamentals partition into five mode-families, indexed by the order of f in the cyclic group $\mathbb{Z}/12$:

- **Full chromatic (period 12).** The four fundamentals coprime to 12: $f \in \{1, 5, 7, 11\}$. Each produces a cycle visiting every note in $\{1, 2, \ldots, 12\}$ exactly once.
  - $f = 1$: the *chromatic scale* itself — notes ascend by one semitone, visiting $1, 2, 3, \ldots, 12$ in order.
  - $f = 5$: the *circle of fourths* — notes ascend by five semitones at each step, visiting $5, 10, 3, 8, 1, 6, 11, 4, 9, 2, 7, 12$.
  - $f = 7$: the *circle of fifths* — notes ascend by seven semitones at each step, visiting $7, 2, 9, 4, 11, 6, 1, 8, 3, 10, 5, 12$. This is the fundamental cycle on which Western tonal harmony is built.
  - $f = 11$: the *descending chromatic* — notes ascend by eleven semitones at each step, which is equivalent to descending by one semitone. The sequence is the chromatic scale walked in reverse.
- **Whole-tone (period 6).** Two fundamentals: $f \in \{2, 10\}$. Each produces a cycle visiting six of the twelve notes — specifically, the even-positioned notes $\{2, 4, 6, 8, 10, 12\}$ (or a translation thereof). This is the *whole-tone scale*, used famously by Debussy and other impressionist composers.
- **Diminished seventh (period 4).** Two fundamentals: $f \in \{3, 9\}$. Each produces a four-note cycle. For $f = 3$, the notes are $\{3, 6, 9, 12\}$, the four positions of a *diminished seventh chord* — equally spaced minor thirds.

- **Augmented triad (period 3).** Two fundamentals: $f \in \{4, 8\}$. Each produces a three-note cycle. For $f = 4$, the notes are $\{4, 8, 12\}$, the three positions of an *augmented triad* — equally spaced major thirds.
- **Tritone (period 2).** One fundamental: $f = 6$. The cycle has two notes: $\{6, 12\}$, an interval of six semitones — the *tritone*, the most dissonant interval in tonal harmony.

The $\varphi(12) = 4$ full-chromatic modes are the cycles that traverse all twelve notes; the remaining seven nontrivial fundamentals produce shorter modes that traverse subsets. The fundamental $f = 12$ produces a degenerate one-note cycle (the unison, just the note 12 repeated) and is not counted as a mode.

Each named cycle has direct musical content. The circle of fifths ($f = 7$) is, plausibly, the most important repeating structure in Western tonal music, organizing key signatures, modulations, and harmonic progressions. The whole-tone scale ($f = 2$) is a recognizable feature of impressionist and modern compositions. The augmented triad and diminished seventh are characteristic chords with long histories in tonal and post-tonal practice. And the tritone has its own famous history as the *diabolus in musica*.

2.3 Modes at Other Bases

The same construction applies at any base $b \geq 3$, and the number of full-cycle modes — those for which the period $T(f, b)$ equals $b - 1$ — is $\varphi(b - 1)$.

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base b	$b-1$	$\varphi(b-1)$	tuning system	full modes
3	2	1	one-tone	only $f = 1$
6	5	4	pentatonic	$f \in \{1, 2, 3, 4\}$
8	7	6	diatonic-sized	$f \in \{1, 2, 3, 4, 5, 6\}$
13	12	4	chromatic	$f \in \{1, 5, 7, 11\}$
20	19	18	19-tone	every $f \in \{1, \ldots, 18\}$
50	49	42	49-tone	f coprime to $49 = 7^2$

A *prime-modulus base* — one with $b - 1$ prime, like $b = 8$ ($b - 1 = 7$) — has $\varphi(b - 1) = b - 2$ full modes, because every nonzero f in $\{1, \ldots, b - 2\}$ is coprime to $b - 1$. A *highly composite-modulus base* like $b = 13$ ($b - 1 = 12$) has fewer full modes ($\varphi(12) = 4$) because many fundamentals share factors with $b - 1$.

A musical takeaway: the *richness* of modal structure at a given base depends inversely on the composite-richness of $b - 1$. The chromatic system at base 13 has only 4 full modes despite 12 available pitches because 12 has many divisors; a 19-tone equal-temperament system at base 20, by contrast, has 18 full modes (every nontrivial fundamental is full) because 19 is prime. Composers experimenting with non-twelve-tone tunings might find this an organizing principle.

2.4 What the Musical Reading Shows

The musical reading of the harmonic-series specialization is the first of the paper's four exhibits, and it does what the others will do as well: take a structural arithmetic fact (here, the period $(b - 1)/\gcd(f, b - 1)$ from [2]) and show that, at the right interpretation of the base, it has direct content of a recognizable kind. The fundamentals coprime to $b - 1$ are the multiplicative generators of the cyclic group $\mathbb{Z}/(b - 1)$; the modes they produce are the orbits of that group acting on itself by multiplication. In music-theoretic language, this is the statement that the four "complete" modes at base 13 (chromatic, circle of fourths, circle of fifths, descending chromatic) are the generators of the cyclic group $\mathbb{Z}/12$. The structural identification is exact, and the musical content is what makes the abstract group structure perceptible — a major scale walked in semitones, a sequence of perfect fifths walked through every key.

The reading also illustrates the more general principle that runs through the four exhibits of this paper: the *base b is a lens*, and different bases reveal different structures. The chromatic system at base 13 reveals certain modes; the diatonic-sized system at base 8 reveals others. The number-theoretic fact $\varphi(b-1)$ governs the modal richness; the musical content emerges from the specific choice of base.

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3 Ramanujan's Partition Congruences as Stripes

3.1 The Partition Function and Ramanujan's Congruences

The *partition function* $p(n)$ counts the number of ways to write the positive integer n as a sum of positive integers, where the order of summands does not matter. The first few values are:

$p(0) = 1,\ \ p(1) = 1,\ \ p(2) = 2,\ \ p(3) = 3,\ \ p(4) = 5,\ \ p(5) = 7,\ \ p(6) = 11,\ \ p(7) = 15,\ \ p(8) = 22,\ \ldots$

The partition function grows rapidly — asymptotically $p(n) \sim \frac{1}{4n\sqrt{3}} \exp(\pi\sqrt{2n/3})$ — and has no closed-form generating function in elementary terms beyond Euler's identity

$$\sum_{n\geq 0} p(n)x^n \;=\; \prod_{k\geq 1} \frac{1}{1-x^k}.$$

In 1919, Srinivasa Ramanujan [4] discovered three remarkable congruences satisfied by the partition function:

$$p(5k+4) \equiv 0 \pmod 5, \quad p(7k+5) \equiv 0 \pmod 7, \quad p(11k+6) \equiv 0 \pmod{11},$$

for every $k \geq 0$. These are the only "elementary" Ramanujan-style congruences with small modulus; no congruence $p(13k+r) \equiv 0 \pmod{13}$ holds for any r, and the failure of the pattern at modulus 13 was one of Ramanujan's most famous early results.

The three congruences have been generalized in various ways — to modular forms, to congruences modulo $5^k, 7^k, 11^k$, and to congruences involving other modular invariants — but the three classical ones at moduli $5, 7, 11$ remain the most visible and the most easily stated.

3.2 Stripes at Bases 6, 8, 12

In the octave representation, Ramanujan's congruences become directly visible as a regular stripe pattern in the note signal of the partition function at three specific bases.

> **Proposition (Ramanujan Stripes).** Let $b \geq 6$ and consider the note signal $\mathrm{note}(p(n), b)$ of the partition function at base b.
>
> - At $b = 6$ $(b-1 = 5)$, for every $k \geq 0$ and $n = 5k+4$, the note value is $\mathrm{note}(p(n), 6) = 5$.

- At $b = 8$ $(b-1 = 7)$, for every $k \geq 0$ and $n = 7k+5$, the note value is $\mathrm{note}(p(n), 8) = 7$.
- At $b = 12$ $(b-1 = 11)$, for every $k \geq 0$ and $n = 11k+6$, the note value is $\mathrm{note}(p(n), 12) = 11$.

Proof. In each case, Ramanujan's congruence asserts that $p(n)$ is divisible by $b-1$ at the specified arithmetic progression of n. By the definition of the note function,

$$\mathrm{note}(N, b) = 1 + ((N-1) \bmod (b-1)),$$

we have $\mathrm{note}(N, b) = b-1$ if and only if $N - 1 \equiv b-2 \pmod{b-1}$, equivalently $N \equiv 0 \pmod{b-1}$. Substituting $N = p(n)$ at the specified n, the Ramanujan congruence guarantees this. $\square$

The signal at base 6 for the first 20 values of $p(n)$:

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n	$p(n)$	$\mathrm{note}(p(n), 6)$	Ramanujan position?
0	1	1	
1	1	1	
2	2	2	
3	3	3	
4	5	**5**	$(n = 5 \cdot 0 + 4)$
5	7	2	
6	11	1	
7	15	**5**	
8	22	2	
9	30	**5**	$(n = 5 \cdot 1 + 4)$
10	42	2	
11	56	1	
12	77	2	
13	101	1	
14	135	**5**	$(n = 5 \cdot 2 + 4)$
15	176	1	
16	231	1	
17	297	2	
18	385	**5**	
19	490	**5**	$(n = 5 \cdot 3 + 4)$

The signal exhibits a guaranteed 5 at every Ramanujan position ($n = 4, 9, 14, 19, \ldots$), interspersed with other 5s and other notes at the remaining positions. The "stripe" is the *guaranteed* pattern — the regular comb of 5s at every fifth index starting from 4.

A few observations from the table:

- At $n = 7$ and $n = 18$, the note value is also 5, but these are not Ramanujan positions; they arise from $p(7) = 15$ and $p(18) = 385$, both of which happen to be divisible by 5 for reasons not predicted by Ramanujan's congruence (and $p(18) = 385 = 5 \cdot 7 \cdot 11$, in fact divisible by all three Ramanujan moduli). The stripe is a *minimum* — it guarantees 5s at the Ramanujan positions; 5s elsewhere are extra.
- The non-Ramanujan note values at the remaining positions $(1, 2, 3, 4)$ appear with apparent statistical irregularity. This is expected: outside the Ramanujan congruence, the partition

function modulo 5 is not constrained by elementary identities, and its distribution among the residues is governed by deeper questions in the theory of modular forms.

Analogous tables can be constructed at base 8 (where the 7-stripe appears at $n = 5, 12, 19, 26, \ldots$) and at base 12 (where the 11-stripe appears at $n = 6, 17, 28, 39, \ldots$).

3.3 The Structural Reason for the Stripes

The stripe phenomenon is, structurally, the simplest possible application of Theorem 4 of [2]: if a sequence is constrained modulo d at certain indices, the note signal of that sequence at base $d+1$ inherits the constraint at those indices. The constraint in Ramanujan's congruences is that $p(n) \equiv 0 \pmod 5$ at the indices $n \equiv 4 \pmod 5$; the note signal at base 6 inherits this as a stripe at note value 5.

The three stripes correspond to the three Ramanujan congruences, and the three bases $6, 8, 12$ correspond to $5+1, 7+1, 11+1$ — the moduli of the three congruences shifted by one to land at the right base in our framework.

3.4 What the Stripe Exhibit Shows

The Ramanujan congruences are, in the conventional presentation, statements about arithmetic progressions of indices at which the partition function is divisible by a small prime. They are abstract and require careful statement. In the octave representation at the right base, they become visible patterns — regular columns of a fixed note value amid an otherwise irregular signal. The classical theorem is the same; the perceptibility is what changes.

This exhibit, like the musical-modes exhibit of §2, demonstrates the framework's principal use: a classical structural fact is *not* derived in the framework, but the framework provides a coordinate system in which the fact is visually direct. Ramanujan's congruences took the mathematician of 1919 considerable insight to discover and prove. In note coordinates at base 6, they are just stripes — anyone plotting the signal sees them. The discovery is not made by the representation, but the *recognition* is.

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4 Catalan Parity as a Powers-of-2 Indicator

The shortest of the four exhibits showcases a classical parity rule about the Catalan numbers: C_n is odd exactly when $n+1$ is a power of 2. At base $b = 3$ — the smallest non-degenerate base — this rule manifests as a direct signal whose note value 1 marks precisely the indices $n = 2^k - 1$, and whose note value 2 marks all the others.

4.1 The Catalan Numbers and the Parity Rule

The *Catalan numbers* C_n count an extraordinary number of distinct combinatorial structures: triangulations of an $(n+2)$-gon, Dyck paths of length $2n$, balanced strings of n matched parentheses, full binary trees with $n+1$ leaves, and dozens of others. The closed form is

$$C_n \;=\; \frac{1}{n+1}\binom{2n}{n} \;=\; \binom{2n}{n} - \binom{2n}{n-1},$$

and the first few values are

$$C_0 = 1,\ C_1 = 1,\ C_2 = 2,\ C_3 = 5,\ C_4 = 14,\ C_5 = 42,\ C_6 = 132,\ C_7 = 429,\ C_8 = 1430,\ \ldots$$

A striking parity property holds:

> **Parity Rule.** For every $n \geq 0$, the Catalan number C_n is odd if and only if $n+1$ is a power of 2.

That is, C_n is odd exactly when $n \in \{0, 1, 3, 7, 15, 31, 63, 127, \ldots\}$ — the *Mersenne-like* positions, one less than powers of 2. At every other index, C_n is even.

4.2 Proof of the Parity Rule

The proof uses Kummer's theorem on the p-adic valuation of binomial coefficients: the largest power of a prime p dividing $\binom{m+n}{n}$ equals the number of carries when m and n are added in base p.

Apply this with $p = 2$ to $\binom{2n}{n}$. The number of carries when computing $n + n$ in base 2 equals the number of 1-bits in the binary expansion of n: each 1-bit doubles to a carry on the next position. Writing $s_2(n)$ for the binary digit-sum,

$$v_2\left(\binom{2n}{n}\right) \;=\; s_2(n),$$

where $v_2(\cdot)$ denotes the 2-adic valuation.

Now $C_n = \binom{2n}{n}/(n+1)$, so

$$v_2(C_n) \;=\; s_2(n) - v_2(n+1).$$

For C_n to be odd, $v_2(C_n) = 0$, which requires $s_2(n) = v_2(n+1)$.

Write $n + 1 = 2^a \cdot m$ with m odd and $a \geq 0$. Then $v_2(n+1) = a$ and $n = 2^a m - 1$. The binary expansion of n is computed as

$$n \;=\; 2^a m - 1 \;=\; 2^a(m-1) + (2^a - 1),$$

which in binary consists of a ones in the low-order positions (from $2^a - 1$) followed by the binary expansion of $m - 1$ shifted up by a positions. Therefore

$$s_2(n) \;=\; a + s_2(m-1).$$

Substituting: $s_2(n) = v_2(n+1)$ becomes $a + s_2(m-1) = a$, so $s_2(m-1) = 0$, so $m - 1 = 0$, so $m = 1$. This gives $n + 1 = 2^a$ — a power of 2.

Conversely, if $n + 1 = 2^a$, then $m = 1$ and $s_2(m-1) = 0$, so $s_2(n) = a = v_2(n+1)$ and $v_2(C_n) = 0$ — meaning C_n is odd. $\square$

4.3 The Note Signal at Base 3

At base $b = 3$, the note alphabet is $\{1, 2\}$, and the note function is simply

$$\text{note}(N, 3) \;=\; \begin{cases} 1 & \text{if } N \text{ is odd,} \\ 2 & \text{if } N \text{ is even.} \end{cases}$$

The note signal of the Catalan numbers at base 3 is therefore the parity sequence of (C_n) in note-coordinate form.

By the Parity Rule, $\mathrm{note}(C_n, 3) = 1$ exactly when $n+1$ is a power of 2, and $\mathrm{note}(C_n, 3) = 2$ otherwise. The signal is a *powers-of-2 indicator*: it visits 1 at the indices $n = 0, 1, 3, 7, 15, 31, 63, \ldots$, and visits 2 at every other index.

The first values of the signal:

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n	C_n	$\mathrm{note}(C_n, 3)$	$n+1$ is a power of 2?
0	1	**1**	$(n+1=1)$
1	1	**1**	$(n+1=2)$
2	2	2	
3	5	**1**	$(n+1=4)$
4	14	2	
5	42	2	
6	132	2	
7	429	**1**	$(n+1=8)$
8	1430	2	
$\vdots$	$\vdots$	$\vdots$	
14	2674440	2	
15	9694845	**1**	$(n+1=16)$
16	35357670	2	
$\vdots$	$\vdots$	$\vdots$	
30	3814986502092304	2	
31	14544636039226909	**1**	$(n+1=32)$

The signal is a sequence of pulses (the 1s) at exponentially-spaced positions amid a sea of background (the 2s). The pulses are sparse: between $n = 7$ and $n = 15$, the signal is uniformly 2; between $n = 15$ and $n = 31$, the signal is uniformly 2; the gaps between consecutive pulses approximately double each time.

4.4 What the Indicator Exhibit Shows

The Catalan-parity exhibit is the cleanest of the four in this paper. The base is small ($b = 3$, the smallest base our framework admits), the note alphabet is just $\{1, 2\}$, and the rule is the simplest possible: pulse at positions $2^k - 1$, background everywhere else.

What the exhibit shows is that even a classical parity rule — Kummer-and-Catalan, well-known to combinatorialists since at least the mid-twentieth century — has a perceptible representation as a discrete signal under the octave-representation lens. The information content is the same as in the original combinatorial statement; the form is just easier to read off and easier to relate to other indicator-style signals in the literature.

The base 3 specifically is the right base because the underlying constraint ("C_n is odd") is a modulo-2 constraint, and the note alphabet at base 3 has $b - 1 = 2$ values — exactly the parity alphabet. The same construction at any other base would lose the clean two-valued interpretation, but more refined congruence rules (C_n mod 3, C_n mod 5) would emerge at other bases. We do not pursue them here.

5 The Oliver–Soundararajan Prime Last-Digit Bias

5.1 Dirichlet Equidistribution and the Expected Distribution

Dirichlet's theorem on primes in arithmetic progressions (1837) states that for any modulus $m \geq 2$ and any residue class a coprime to m, there are infinitely many primes $p \equiv a \pmod{m}$. A stronger version (Dirichlet's density theorem) holds: the primes are equidistributed among the residue classes coprime to m, with each class containing $1/\varphi(m)$ of the primes in the long run.

Specializing to $m = 10$: the residue classes coprime to 10 are $\{1, 3, 7, 9\}$, with $\varphi(10) = 4$. So in the long run, each of these four last-digit classes contains exactly 1/4 of the primes. The expected probability that a randomly-chosen prime has any specific last digit in $\{1, 3, 7, 9\}$ is 1/4.

A natural question follows: what is the conditional probability that the *next* prime has a specific last digit, given the current prime's last digit? If the last digits of consecutive primes were independent, the joint distribution would be uniform, and the conditional probability $\Pr(p_{n+1} \bmod 10 = a \mid p_n \bmod 10 = b)$ would equal the unconditional probability 1/4.

For a long time, this was the expected answer: consecutive prime last digits are independent, governed by Dirichlet equidistribution. The Oliver–Soundararajan result of 2016 showed that this expectation is wrong.

5.2 The Oliver–Soundararajan Result

In 2016, Robert Lemke Oliver and Kannan Soundararajan [5] proved (conditionally on a strong form of the Hardy–Littlewood prime k-tuples conjecture) and verified computationally to a high degree of confidence that:

> **The Oliver–Soundararajan Bias.** For consecutive primes p_n, p_{n+1} and any residue class a coprime to 10, the conditional probability $\Pr(p_{n+1} \equiv a \pmod{10} \mid p_n \equiv a \pmod{10})$ is *strictly less than* the unconditional probability 1/4.

The bias is small but non-trivial: numerically, the conditional probability that two consecutive primes share the same last decimal digit (computed across the first 10^8 primes) is approximately 0.18 rather than the uniform-expected 0.25 — a deficit of roughly 28% relative to the uniform expectation.

The Oliver–Soundararajan paper provides a detailed heuristic explanation rooted in the Hardy–Littlewood conjecture: the bias arises from a careful counting of admissible gap configurations between consecutive primes, and the specific structure of the modulus 10 enters through the distribution of admissible gaps modulo 10.

The result generalized: consecutive primes also avoid sharing residue classes at other moduli, with analogous deficits at moduli $3, 7, 11$, and beyond.

5.3 The Bias in Note Coordinates at Base 11

At base $b = 11$, the note signal of a prime $p > 5$ is directly the last decimal digit of p. From the note formula

$$\mathrm{note}(p, 11) \; = \; 1 + ((p - 1) \bmod 10),$$

for $p \equiv a \pmod{10}$ with $a \in \{1, 3, 7, 9\}$, the note value is $\mathrm{note}(p, 11) = a$. Explicitly:

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last digit of p	$\mathrm{note}(p, 11)$
1	1
3	3
7	7
9	9

The note signal of the prime sequence at base 11 is therefore the sequence of last digits of consecutive primes, in note-coordinate form. (The primes $p = 2$ and $p = 5$ give note values 2 and 5 respectively, outside the $\{1, 3, 7, 9\}$ sub-alphabet; we ignore them in what follows, as is conventional for the Dirichlet-equidistribution analysis.)

The Oliver–Soundararajan bias is therefore directly observable in this signal: consecutive note values are *less likely to coincide* than the 1/4 uniform-distribution prediction would suggest. In particular, the joint distribution of consecutive notes $(\mathrm{note}(p_n, 11), \mathrm{note}(p_{n+1}, 11))$ exhibits a measurable deficit on the diagonal (the four pairs $(1, 1), (3, 3), (7, 7), (9, 9)$) relative to the off-diagonal pairs.

A small table of the first 20 primes $p > 5$ and their notes at base 11:

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n	p_n	$\mathrm{note}(p_n, 11)$
1	7	7
2	11	1
3	13	3
4	17	7
5	19	9
6	23	3
7	29	9
8	31	1
9	37	7
10	41	1
11	43	3
12	47	7
13	53	3
14	59	9
15	61	1
16	67	7
17	71	1
18	73	3
19	79	9
20	83	3

Across these 19 consecutive prime pairs, there is no pair $(\mathrm{note}(p_n, 11), \mathrm{note}(p_{n+1}, 11))$ with both notes equal — no immediate repeats. This is a small-sample illustration of the bias, but the same trend persists in the long run: consecutive notes match less frequently than 1/4 of the time.

5.4 The Bias at Other Bases

The Oliver–Soundararajan bias generalizes to other moduli. At any base b with $b - 1 = m$, the note signal of the prime sequence visits the residues coprime to m in note-coordinate form, and the bias against consecutive matches is detectable at every such base. The specific magnitudes of the bias depend on the modulus m and on the distribution of admissible prime gaps modulo m.

The classical base for studying this bias — base 11, with $m = 10$ — is the natural choice because it corresponds to the familiar base-10 representation of the integers. Other bases correspond to other modular structures: base 13 ($m = 12$, with $\varphi(12) = 4$ coprime residues) shows the bias on residues $\{1, 5, 7, 11\}$; base 8 ($m = 7$, prime) shows the bias on $\{1, 2, 3, 4, 5, 6\}$.

The Oliver–Soundararajan paper [5] gives precise asymptotic predictions for the bias at each modulus, conditional on the Hardy–Littlewood prime k-tuples conjecture. The framework here provides a uniform visual presentation of those biases at different bases — but does not contribute new arithmetic to the analysis.

5.5 What the Bias Exhibit Shows

Like the three preceding exhibits, the Oliver–Soundararajan bias is classical in content: a 2016 result with a detailed heuristic explanation and computational verification. What the octave representation contributes is a coordinate system in which the bias becomes directly observable as a property of consecutive notes — the visual form of the underlying analytic-number-theoretic statement.

Where the Ramanujan stripes were a *guaranteed* pattern (Ramanujan's congruences are theorems) and the Catalan parity was a *deterministic* indicator (Kummer's theorem gives a binary yes/no), the Oliver–Soundararajan bias is *statistical* — a deviation from uniformity that emerges over many primes. The signal therefore has a different character: not a single pattern with a closed-form description but a measurable bias visible in the statistics. Yet the framework presents it cleanly: the note signal is the right coordinate system, and the bias appears as a coupling between consecutive values.

The exhibit closes the set of four with a different kind of perceptibility from the others — *statistical* rather than *exact*. The four exhibits together demonstrate the framework's flexibility: deterministic congruences (Ramanujan), deterministic parities (Catalan), structural cycles (musical modes), and now a statistical bias all appear cleanly in note coordinates at the appropriate base.

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6 What the Exhibits Show Together

The four exhibits — interval cycles, Ramanujan stripes, Catalan parity, prime last-digit bias — were chosen from disparate corners of mathematics: music theory, partition theory, combinatorial enumeration, analytic number theory. They share no common subject matter. Each exhibit treats a classical result, and in each case the result was discovered and proved in its own context, with its own techniques, decades or centuries before the octave representation existed.

What unites them is a single structural observation. Each exhibit takes a classical fact about an integer sequence — a periodicity, a divisibility, a parity, a statistical bias — and shows that the *choice of base* in the octave representation is what makes the fact directly perceptible in note coordinates. The arithmetic content does not change. What changes is the visibility.

6.1 The Four Exhibits as One Pattern

Schematically, each exhibit follows the same recipe:

1. Take an integer sequence (a_n) whose modular structure modulo some integer d is governed by a classical theorem (Ramanujan, Kummer, Oliver–Soundararajan, or the elementary harmonic gcd law).
2. Compute the note signal $\mathrm{note}(a_n, b)$ at the base $b = d + 1$.
3. Observe that the modular structure manifests as a direct pattern in the note signal: a stripe (Ramanujan), a pulse train (Catalan), a cyclical mode (musical), or a statistical coupling (Oliver–Soundararajan).

The choice of base $b = d + 1$ is what aligns the framework with the underlying modulus. Picked correctly, the base brings the structural fact to the surface as a visible or audible pattern. Picked incorrectly — at a base whose modulus $b - 1$ does not divide d — the same fact is still present in the data but is obscured by the lens.

6.2 The Free-Parameter Reading

The base b is not fixed by the framework. Different bases reveal different facets of the same integer sequence. At base 13, the harmonic series reveals musical modes; at base 11, the prime sequence reveals the Oliver–Soundararajan bias; at base 3, the Catalan numbers reveal their parity rule; at bases $6, 8, 12$, the partition function reveals Ramanujan's congruences.

This is the central observation of the exhibits — that the choice of base is the choice of *what arithmetic property* of the sequence becomes audible. The companion paper [forthcoming] develops this observation into a methodological framework, the *base as a free parameter* thesis, with a resonance principle for selecting bases tied to arithmetic structures. The exhibits of this paper are the empirical evidence for that thesis.

6.3 The Heterogeneity is the Point

A reader might object that we have picked four exhibits to suit the framework, and that any sufficiently flexible coordinate system would make some classical facts visible at some choice of coordinates. There is force in the objection. We do not claim that the octave representation is unique among coordinate systems in possessing this property.

What we do claim is that the four exhibits are heterogeneous enough that the same framework picking out a perceptible form for all of them is non-trivial. A music-theoretic mode, a partition-theory congruence, a combinatorial-number-theory parity, and an analytic-number-theory bias do not have a natural shared coordinate system in standard mathematical practice. The octave representation supplies one. Whether other coordinate systems do as well, and whether they pick out different structures with the same flexibility, are open questions we do not pursue.

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7 Scope, Novelty, Related Work, Open Questions, and References

7.1 The Contribution

This paper contributes four exhibits — concrete demonstrations that the octave representation of [1], at well-chosen bases, makes classical structural facts about integer sequences directly perceptible. Specifically:

- The harmonic-series specialization of [2] gives a musical mode at each base, with $\varphi(b-1)$ full modes per base. At base 13, the four full modes are the chromatic scale, the circle of fourths, the circle of fifths, and the descending chromatic — the named structural cycles of Western tonal music.
- Ramanujan's three classical partition congruences (1919) appear as guaranteed stripes in the partition note signal at bases $6, 8, 12$ — the bases corresponding to the moduli $5, 7, 11$ in the framework.
- The Catalan parity rule (the classical fact that C_n is odd iff $n+1$ is a power of 2) manifests as a powers-of-2 indicator signal at base 3, with note value 1 at the Mersenne-like indices and note value 2 elsewhere.
- The Oliver–Soundararajan bias (2016) on consecutive prime last digits is directly observable in the prime note signal at base 11, as a statistical deficit in adjacent note matches.

These four exhibits are not theorems; they are demonstrations. Their unifying observation — that base b is a lens whose choice makes underlying modular structure perceptible — sets up the methodological argument of the next paper in the series.

7.2 What is Not Claimed

Every arithmetic fact reported in the exhibits is classical.

- The **harmonic-series period formula** from [2] (which underlies the musical modes) is elementary; the musical interpretation as named modes is implicit in standard music theory.
- **Ramanujan's partition congruences** are due to Ramanujan [4] (1919). The general theory of partition congruences was developed in the twentieth century by Watson, Atkin, Ono, and others, and is now framed in the language of modular forms.
- **The Catalan parity rule** is a standard exercise; the proof via Kummer's theorem appears in textbooks (Stanley [6], Sloane's OEIS, and others).
- **The Oliver–Soundararajan bias** is due to Lemke Oliver and Soundararajan [5] (2016), with extensive subsequent computational and heuristic follow-up.
- **Dirichlet's equidistribution** is due to Dirichlet (1837).
- **Kummer's theorem** on the p-adic valuation of binomial coefficients is due to Kummer (1852).

What is new in this paper is not the arithmetic but the unified presentation: the demonstration that a single representation — the octave representation of [1] — makes all four of these heterogeneous classical results directly perceptible at the appropriate base.

7.3 Related Work

The relationship between music and mathematics, particularly the interaction of cyclic groups with musical structure, has a substantial literature. The musical-modes exhibit of §2 is essentially the statement that the cyclic group $\mathbb{Z}/12$ generates the named scales of Western tonal music under multiplication; this is implicit in any standard music-theory text and is made explicit in mathematical music theory (Lewin's transformational theory, Forte's pitch-class set theory). The framework here adds the observation that the same construction generalizes uniformly to all bases, with the modal richness governed by $\varphi(b-1)$.

The theory of partition congruences and their modular-forms interpretation is a major area of modern number theory. The exhibits in §3 do not contribute to that theory; they merely present its most classical statements (Ramanujan's three congruences) in a different coordinate system.

The Catalan parity rule and its many generalizations (binomial-coefficient parities under Kummer's theorem, p-adic structure of generating functions, etc.) form their own subliterature in combinatorial number theory. We have selected the simplest case (parity at base 3) as the cleanest exhibit.

The Oliver–Soundararajan paper sparked considerable subsequent work on biases in prime distributions, both at last-digit moduli and at other moduli. The note-signal presentation at base 11 is, to our knowledge, novel as a uniform-presentation device, but the underlying bias has been studied in considerable depth in the original setting.

7.4 Open Questions

The exhibits-driven format of this paper makes its open questions different in kind from those of the theorem-driven papers in the series.

1. **Further exhibits.** The four exhibits selected here are representative but not exhaustive. Other classical congruences — Wilson's, Bell-number Touchard's, the Schur partition identities — might admit analogous note-coordinate presentations at well-chosen bases. We have not pursued these.
2. **Negative results.** Are there classical structural facts about integer sequences whose modular structure does *not* admit a clean note-coordinate presentation? The framework is general enough that we expect very few such facts, but a systematic study has not been undertaken.
3. **The base-as-lens program.** The exhibits motivate the methodological argument of the companion paper [forthcoming], which develops the "base as a free parameter" thesis and a resonance principle for matching bases to arithmetic structures. The present paper is the empirical groundwork; the methodological argument is the framework's natural next step.

7.5 References

[1] Carlson, A. (2026). *The Octave Representation: A Musical Re-coordinatization of Positional Number Systems.* https://adamcarlson.io/papers/octave-representation.pdf

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[4] Ramanujan, S. (1919). *Some properties of $p(n)$, the number of partitions of n.* Proceedings of the Cambridge Philosophical Society 19, 207–210.

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