# The Golden Recursion Tower and its Unique Geometric Embedding

Coupled Epicycles in $\mathbb{R}^4$, the Golden Ratio at Every Level, and Why $N = 2$ is Privileged

Adam Carlson
me@adamcarlson.io

June 2026

## 1 Introduction

The golden ratio $\varphi = (1 + \sqrt{5})/2$ is most often introduced as the positive root of $x^2 = x + 1$, equivalently the limit of ratios of consecutive Fibonacci numbers. Many of its appearances in mathematics — quasicrystals, KAM tori, continued fractions, phyllotaxis — turn on a single algebraic fact: $\varphi$ is the unique positive direction preserved by a specific substitution. Begin with any pair of positive numbers $(R, r)$ and apply the substitution $(R, r) \mapsto (r, R - r)$. The pair gets remapped to a new pair. The original direction $(R, r)$ is preserved (up to scale) by this substitution if and only if $R/r = \varphi$, in which case the substitution acts as multiplication by $1/\varphi$. Every other ratio is moved off its line; only $\varphi$ stays put.

This algebraic fact is the engine of what follows. It has, we will argue, a *geometric* meaning that is largely unremarked-on: there is a natural surface in $\mathbb{R}^4$ — built as a sum of two classical epicycles — on which the substitution above acts as uniform scaling precisely when the radius ratio is $\varphi$. The surface is well-defined for any positive ratio, but it admits a smooth embedding into $\mathbb{R}^4$ only above a phase boundary, and the substitution-as-scaling characterization picks out $\varphi$ from this embedding regime as the unique algebraic-recursion fixed point.

So far this is local information about one specific construction. The first surprise — and the central content of this paper — is that the construction generalizes upward. There is a natural $N$-level extension of the surface, an iterated coupled-epicycle in higher dimensions, and the same characteristic equation $c^2 + c - 1 = 0$ that picks out $\varphi$ at level 2 continues to pick it out at every level $N \geq 2$. The characteristic equation does not change with $N$. The golden ratio is the unique amplitude direction preserved by the natural substitution at every level of the tower, with the same scaling factor $1/\varphi$ at every level. This is what we mean by the *Golden Recursion Tower*: an infinite algebraic structure in which the golden ratio appears, at every level, in the same role and for the same reason.

The second surprise — and the structural punchline of the paper — is that the geometric realization of this tower in $\mathbb{R}^4$ exists at only one level. At $N = 2$ the surface is a smooth 2-torus embedded in $\mathbb{R}^4$, the geometry has closed-form curvature invariants involving $\varphi$, the embedding is provable, and the geodesic flow is chaotic. At $N \geq 3$ the natural $\mathbb{R}^4$ realization is *structurally singular*: the Jacobian of the parameterization drops rank at $2^N$ diagonal corner points of $T^N$, independent of the radii chosen, for reasons that we will see are baked into the construction. The two-level torus is the *unique* clean geometric instance, in $\mathbb{R}^4$, of a tower that exists algebraically at every level.

These two facts together — an algebraic tower governed by a single characteristic equation, and a geometric realization that exists at only one level of that tower — are what gives the title of the paper. *The* Golden Recursion Tower is the algebraic object. *Its* unique geometric embedding is the two-level torus. The paper develops both.

The technical content rests on two theorems. The **Universal Recursion Theorem** (Theorem 1, §4) states that for every $N \geq 2$, the natural substitution at level $N$ is realized as uniform scaling by $1/\varphi$ if and only if the amplitude sequence $(R_0, R_1, \ldots, R_{N-1})$ has consecutive ratios $R_n/R_{n+1} = \varphi$ throughout. The proof is a single-line characteristic-equation argument that does not depend on $N$. The **Singularity Theorem** (Theorem 2, §5) states that for every $N \geq 3$, the natural codimension-1 realization $\gamma_N : T^N \to \mathbb{R}^4$ has rank $\leq 2$ Jacobian at every diagonal corner of $T^N$, for any choice of radii. The proof rests on the observation that all sines vanish at diagonal corners, two of the four Jacobian rows are therefore identically zero, and the rank cannot exceed two regardless of the radii multiplying nonzero entries. The two theorems together produce the title's claim: the algebraic tower exists at every level but admits clean codim-2 embedding in $\mathbb{R}^4$ only at $N = 2$.

The remainder of the paper is organized as follows. Section 2 sets up the two-level case and recasts the construction as a sum of two classical Ptolemy epicycles in $\mathbb{R}^4$, distinguished by a specific phase coupling between the two coordinate planes. Section 3 establishes the substitution recursion at level 2 and proves the original $\varphi$ characterization (Lemma 1, with its one-line proof). Section 4 extends to arbitrary $N$ and proves the Universal Recursion Theorem. Section 5 treats the geometric realization in $\mathbb{R}^4$ and proves the Singularity Theorem, with a discussion of why $N = 2$ is privileged by ambient dimension considerations. Section 6 collects what is known about the two-level surface as a Riemannian object: closed-form curvature extrema involving $\varphi$, a smooth embedding theorem, and the chaotic character of the geodesic flow. Section 7 closes with the algebra-vs-geometry framing and the open questions.

---

# 2 The Coupled Epicycle Construction

## 2.1 Classical epicycles, briefly

A classical Ptolemaic epicycle is a sum of two circular motions in a plane. Identifying the plane with $\mathbb{C}$, an epicycle takes the form

$$\zeta(t) \;=\; R\, e^{i\theta_1} + r\, e^{i\theta_2}, \qquad \theta_1, \theta_2 \in [0, 2\pi),$$

which is, when $\theta_1$ and $\theta_2$ vary independently, the image of the torus $T^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$ in $\mathbb{C}$. If $\theta_2$ is constrained to be a linear function of $\theta_1$, this is a single curve — the deferent–epicycle composition Ptolemy used to model planetary motion. Allowed to vary independently, it traces out a closed annular region of the plane.

Equivalent algebraic form: pulling out the deferent phase,

$$\zeta \;=\; e^{i\theta_1}(R + r\, e^{i\theta_2})\,.$$

The deferent has radius $R$ (the carrier circle) and the epicycle has radius $r$ (a smaller circle whose center is on the deferent). The pair $(\theta_1, \theta_2)$ controls, respectively, where the center of the epicycle sits on the deferent and where the marked point sits on the epicycle. This is the building block of the entire paper.

### 2.2 Two coupled epicycles in

We now place two such epicycles into the two perpendicular complex planes of $\mathbb{R}^4 = \mathbb{C} \oplus \mathbb{C}$. Identify $\mathbb{R}^4$ with $\mathbb{C}^2$ by sending $(w, x, y, z) \in \mathbb{R}^4$ to $(\zeta_1, \zeta_2) = (w + ix, y + iz) \in \mathbb{C}^2$. Define the *coupled epicycle* $\gamma : T^2 \to \mathbb{R}^4$ by specifying its two complex components:

$$\zeta_1 \;=\; e^{i\theta_1}(R + r\,e^{i\theta_2}), \qquad \zeta_2 \;=\; e^{i\theta_2}(R + r\,e^{i(\theta_1 - 2\theta_2)})\,.$$

Equivalently, expanding into real coordinates:

$$\begin{aligned} w &= R\cos\theta_1 + r\cos(\theta_1 + \theta_2), \qquad & x &= R\sin\theta_1 + r\sin(\theta_1 + \theta_2),\\ y &= R\cos\theta_2 + r\cos(\theta_1 - \theta_2), \qquad & z &= R\sin\theta_2 + r\sin(\theta_1 - \theta_2). \end{aligned}$$

The two complex planes of $\mathbb{R}^4$ each carry a classical epicycle. In the $(w, x)$-plane, the deferent is the circle of radius $R$ swept by $\theta_1$, with an epicycle of radius $r$ whose phase relative to the carrier is $\theta_2$. In the $(y, z)$-plane, the deferent is the circle of radius $R$ swept by $\theta_2$, with an epicycle whose phase relative to that carrier is $\theta_1 - 2\theta_2$. The two epicycles are *coupled* in that both share the same pair $(\theta_1, \theta_2)$ as their two angular parameters, but they use these parameters in different roles.

The phase coupling between the two planes can be made explicit. Subtracting off the deferent phase from each, the secondary phases are

$$(\theta_1 + \theta_2) - \theta_1 = \theta_2 \quad \text{in the first plane,} \qquad (\theta_1 - \theta_2) - \theta_2 = \theta_1 - 2\theta_2 \quad \text{in the second.}$$

The angular content of the secondary phases — $(\theta_1 + \theta_2, \theta_1 - \theta_2)$ — is the Hadamard-like linear combination of $(\theta_1, \theta_2)$. This is the source of the construction's distinctive structure: the two epicycles share their angular parameters but combine them by addition in one plane and by subtraction in the other.

### 2.3 What this is, and what it is not

The surface $G = \gamma(T^2) \subset \mathbb{R}^4$ is a 2-torus immersed (and, as we will see, embedded for $R > r\sqrt{2}$) in $\mathbb{R}^4$. It is *not*, despite superficial resemblance, the Clifford torus. The Clifford torus has $R = r$ (equal radii), zero Gaussian curvature everywhere (it is flat), and integrable geodesic flow; our surface has $R \neq r$ as its central feature, mixed-sign curvature, and chaotic geodesic flow. The Clifford torus admits a single-circle interpretation in $S^3 \subset \mathbb{R}^4$; our surface lives in $\mathbb{R}^4$ proper, has no canonical inclusion in any sphere, and is built by summing two epicycles rather than by sweeping a single circle. The radii asymmetry is essential. The construction is a *coupled-epicycle compound*, not a Clifford-type product.

The decomposition $\gamma = A + B$ with

$$A(\theta_1, \theta_2) = R(\cos\theta_1, \sin\theta_1, \cos\theta_2, \sin\theta_2),$$

$$B(\theta_1, \theta_2) = r(\cos(\theta_1 + \theta_2), \sin(\theta_1 + \theta_2), \cos(\theta_1 - \theta_2), \sin(\theta_1 - \theta_2))$$

makes the construction's compound nature explicit. Both $A$ and $B$ are Clifford-torus-type parameterizations — each lies on a 3-sphere in $\mathbb{R}^4$ of constant radius (radius $R\sqrt{2}$ for $A$, radius $r\sqrt{2}$ for $B$) — but $A$ uses the angular parameters directly while $B$ uses their Hadamard-coupled combinations. The sum $\gamma = A + B$ is what produces the coupled-epicycle structure.

# 3 The Substitution Recursion and the Golden Ratio

## 3.1 A substitution on radius pairs

Consider the substitution

$$M : (R, r) \;\mapsto\; (r,\; R - r)$$

acting on the pair of radii. Algebraically, $M$ is the linear map

$$M = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.$$

This is a natural operation in its own right (it appears in Fibonacci-style recurrences and in continued-fraction theory) and is independent of any geometric construction. The question we ask is: for which positive ratios $R/r$ does $M$ act as uniform scaling — that is, $M(R, r) = c \cdot (R, r)$ for some scalar $c$?

## 3.2 The characterization

**Lemma 1.** Let $(R, r)$ be a pair of positive real numbers. The substitution $M(R, r) = (r, R - r)$ equals $c \cdot (R, r)$ for some scalar $c$ if and only if $R/r = \varphi = (1 + \sqrt{5})/2$. The scalar is $c = 1/\varphi$.

*Proof.* The equation $M(R, r) = c(R, r)$ reads, component-wise, $r = cR$ and $R - r = cr$. From the first, $c = r/R$. Substituting into the second: $R - r = (r/R) \cdot r = r^2/R$, so $R^2 - Rr = r^2$, i.e., $R^2 = Rr + r^2$. Dividing by $r^2$ and setting $\rho = R/r$ gives $\rho^2 = \rho + 1$, whose positive solution is $\rho = \varphi$. At this $\rho$, the scalar is $c = r/R = 1/\varphi$. □

This is the algebraic fact mentioned in the introduction. It says nothing yet about geometry.

## 3.3 The geometric meaning

The geometric content emerges when we ask what the substitution does to the *surface* $\gamma$. The surface depends on the radii: write $\gamma_{R,r}$ to make the dependence explicit. We can apply $M$ to the pair $(R, r)$ to get a new pair $(r, R - r)$, and we can ask how $\gamma_{r,R-r}$ relates to $\gamma_{R,r}$.

By direct computation: the new components are

$$\zeta_1^{\text{new}} = e^{i\theta_1}(r + (R - r)e^{i\theta_2}), \qquad \zeta_2^{\text{new}} = e^{i\theta_2}(r + (R - r)e^{i(\theta_1 - 2\theta_2)}).$$

At $R/r = \varphi$, the new pair $(R^{\text{new}}, r^{\text{new}}) = (r, R-r) = (1, 1/\varphi)$ (taking $R = \varphi, r = 1$ as normalization), which is $(1/\varphi)$ times the original $(\varphi, 1)$. By linearity of $\gamma$ in the radii, the new surface is precisely

$$\gamma_{r,R-r} \;=\; \frac{1}{\varphi}\,\gamma_{R,r}\,.$$

At $\varphi$, the algebraic substitution $M$ is realized geometrically as *uniform scaling of the surface in* $\mathbb{R}^4$ *by the factor* $1/\varphi$. At any other ratio, the substitution produces a surface with a different shape (different ratio of radii); only at $\varphi$ is the shape preserved and the substitution reduces to a scaling.

**Theorem 1 (Two-Level Recursion).** At radius ratio $R/r = \varphi$, the substitution $M(R, r) = (r, R - r)$ is realized geometrically as uniform scaling of $\gamma_{R,r}$ by $1/\varphi$:

$$\gamma_{r,R-r} = \frac{1}{\varphi}\,\gamma_{R,r}\,.$$

This is the unique positive ratio at which $M$ is realized as scaling.

This is the formal version of the statement "$\varphi$ is the recursion fixed-point" at the two-level case. The substitution acts on the pair of radii; at $\varphi$, it acts on the surface as scaling; the algebra and the geometry are tied at this one ratio. We will see in §4 that this is not specific to two levels.

---

# 4 The Algebraic Tower

## 4.1 The $N$-level construction

The two-level construction admits a natural extension. We add a third level — an epicycle on top of the epicycle — and continue. The $N$-level construction takes radii $(R_0, R_1, \ldots, R_{N-1})$ and produces a parameterization $\gamma_N : T^N \to \mathbb{R}^4$:

$$\begin{aligned}\zeta_1 &= R_0 e^{i\theta_1} + R_1 e^{i(\theta_1+\theta_2)} + R_2 e^{i(\theta_1+\theta_2+\theta_3)} + \cdots + R_{N-1} e^{i(\theta_1+\theta_2+\cdots+\theta_N)},\\ \zeta_2 &= R_0 e^{i\theta_2} + R_1 e^{i(\theta_1-\theta_2)} + R_2 e^{i(\theta_1-\theta_2+\theta_3)} + \cdots + R_{N-1} e^{i(\theta_1-\theta_2+\theta_3+\cdots+\theta_N)}.\end{aligned}$$

Each successive term adds another level of epicyclic detail, sharing the phase already accumulated by the previous level. At $N = 2$ this is the original construction; at $N = 3$ it adds one term per plane; at general $N$ each plane is a length-$N$ sum.

## 4.2 The generalized substitution

At level $N$, the natural substitution on the radius sequence is

$$M_N : (R_0, R_1, \ldots, R_{N-1}) \;\mapsto\; (R_1, R_2, \ldots, R_{N-1}, R_{N-2} - R_{N-1}).$$

$M_N$ shifts the sequence leftward by one position and produces a new last entry by taking the difference of the previous two. At $N = 2$, this is the original $(R_0, R_1) \mapsto (R_1, R_0 - R_1)$, which is $M$ relabeled. At higher $N$, it is the natural extension.

## 4.3 The universal recursion theorem

**Theorem 2 (Universal Recursion Theorem).** For every $N \geq 2$, the substitution $M_N$ is realized as uniform scaling $M_N(R_0, \ldots, R_{N-1}) = c \cdot (R_0, \ldots, R_{N-1})$ for some scalar $c$ if and only if consecutive ratios satisfy $R_n/R_{n+1} = \varphi$ for every $n = 0, 1, \ldots, N-2$. The scaling factor is $c = 1/\varphi$, independent of $N$.

*Proof.* Suppose $M_N$ acts as uniform scaling by $c$. Component-wise:

$$R_1 = cR_0, \quad R_2 = cR_1, \quad \ldots, \quad R_{N-1} = cR_{N-2}, \quad R_{N-2} - R_{N-1} = cR_{N-1}.$$

The first $N-1$ equations give $R_k = c^k R_0$ for all $k = 0, 1, \ldots, N-1$. The last equation, after substituting these:

$$c^{N-2}R_0 - c^{N-1}R_0 \;=\; c \cdot c^{N-1} R_0 \;=\; c^N R_0.$$

Dividing by $c^{N-2}R_0$ (assuming $c \neq 0$ and $R_0 \neq 0$):

$$1 - c \;=\; c^2, \qquad \text{equivalently} \qquad c^2 + c - 1 = 0.$$

The positive solution is $c = (\sqrt{5} - 1)/2 = 1/\varphi$. The corresponding amplitude ratio is $R_k/R_{k+1} = 1/c = \varphi$ at every adjacent pair. □

The defining equation $c^2 + c - 1 = 0$ is *independent of* $N$. The same equation appears at every level of the tower. This is the universal fact: $\varphi$ is the recursion fixed-point at every level of the multi-level epicycle construction, with the same scaling factor $1/\varphi$ everywhere, and the algebraic mechanism — a single quadratic equation — is the same throughout.

That is the algebraic side of the result. The next question is what happens geometrically.

---

# 5 The Singularity Theorem

## 5.1 A natural expectation, and where it fails

At the two-level case, the surface $\gamma_{R,r}$ embeds smoothly into $\mathbb{R}^4$ for every $R/r > \sqrt{2}$. We proved (§3) that the substitution $M$ is realized geometrically as scaling exactly at $\varphi$, which is comfortably above $\sqrt{2}$ in the embedding regime. The algebraic recursion and the clean geometric embedding co-exist.

The natural expectation is that this co-existence continues at higher levels. Apply the same multi-level construction with radii $(R_0, R_1, \ldots, R_{N-1}) = (\varphi^{N-1}, \varphi^{N-2}, \ldots, 1)$ — the recursion-fixed-point sequence guaranteed by Theorem 2 — and one might hope that $\gamma_N$ embeds cleanly into $\mathbb{R}^4$.

This expectation fails at every $N \geq 3$, in a structural way that is independent of the radii.

## 5.2 The singularity theorem

**Theorem 3 (Singularity Theorem).** For every $N \geq 3$, the parameterization $\gamma_N : T^N \to \mathbb{R}^4$ has Jacobian rank at most 2 at every diagonal corner $(\theta_1, \theta_2, \ldots, \theta_N) \in \{0, \pi\}^N$, independently of the choice of radii $(R_0, R_1, \ldots, R_{N-1})$.

*Proof.* Consider the corner $(\theta_1, \theta_2, \ldots, \theta_N) = (0, 0, \ldots, 0)$. All sines of zero are zero, and all phases in the parameterization at this point are linear combinations of zeros, hence zero. Consequently, every sin(phase) in the formulas for $\partial\gamma_w/\partial\theta_k$ and $\partial\gamma_y/\partial\theta_k$ evaluates to zero, for every $k$.

Concretely: the first component of $\gamma_N$ is $\gamma_w = \sum_{k=0}^{N-1} R_k \cos(\text{phase}_k)$, so

$$\frac{\partial\gamma_w}{\partial\theta_j}(0, \ldots, 0) \;=\; -\sum_k R_k \sin(\text{phase}_k|_{\theta=0}) \cdot \frac{\partial\, \text{phase}_k}{\partial\theta_j}\,.$$

Each $\sin(\text{phase}_k|_{\theta=0}) = \sin(0) = 0$, so the entire sum vanishes, for every $j$. The first row of the Jacobian is therefore identically zero. The same argument applies to the third row, $\partial\gamma_y/\partial\theta_j$.

The Jacobian is a $4 \times N$ matrix. Two of its four rows are identically zero at the diagonal corner. The rank cannot exceed two. The radii $R_k$ multiply zero in every term of the zero rows, so the rank ceiling holds regardless of the radius values.

The argument is identical at every corner $(\epsilon_1\pi, \epsilon_2\pi, \ldots, \epsilon_N\pi)$ with $\epsilon_j \in \{0, 1\}$. There are $2^N$ such corners, and at every one the Jacobian rank is at most two. □

## 5.3 Why $N = 2$ is privileged

The codimension matters. At $N = 2$, the parameterization $\gamma_2 : T^2 \to \mathbb{R}^4$ has $T^2$ of dimension 2 and $\mathbb{R}^4$ of dimension 4, giving codimension 2. The Jacobian is $4 \times 2$, and the immersion condition is rank 2. The argument above still shows that at the corner $(0,0)$ two of the four Jacobian rows are zero — but the matrix has only two columns, so a rank of two is still possible if the remaining two nonzero rows span a two-dimensional subspace. This does happen at $R/r > \sqrt{2}$, and the smooth embedding theorem at the two-level case is provable.

At $N = 3$, the parameterization is $T^3 \to \mathbb{R}^4$, codimension 1. The Jacobian is $4 \times 3$, and the immersion condition is rank 3. The argument forces two zero rows, so the maximum possible rank is 2. The immersion condition fails by an entire rank.

At $N \geq 4$, the parameterization is $T^N \to \mathbb{R}^4$, codimension $4 - N \leq 0$. The map is no longer even an *immersion of* $T^N$ *into* $\mathbb{R}^4$ generically — the image is constrained to be at most four-dimensional, and the rank-2 corner singularities compound onto an already insufficient ambient dimension.

The two-level case is therefore the *unique level at which the natural* $\mathbb{R}^4$ *realization of the recursion tower has the correct codimension for a clean embedding to exist.* Theorem 2 places the algebra at every level; Theorem 3 confines the geometry to one. The combination — algebra-everywhere with geometry-only-at-$N = 2$ — is the structural picture.

### 5.4 Higher ambient dimensions

A natural escape is to allow more ambient dimension at higher levels. The most direct route is $\mathbb{R}^{2N}$, in which each level contributes its own complex plane. Whether the natural “three-plane Hadamard” extension at $N = 3$ in $\mathbb{R}^6$ embeds cleanly is not addressed by this paper. The result we have is about the codimension-1 realization in $\mathbb{R}^4$, and for that specific embedding the singularity at $N \geq 3$ is unavoidable.

---

## 6 The Two-Level Surface

The unique level at which the algebraic tower lands cleanly in $\mathbb{R}^4$ is $N = 2$. The geometry of this case is computable in closed form and has structure tied to $\varphi$ in concrete ways. We collect the salient facts.

### 6.1 The metric

Fixing $R = \varphi, r = 1$, the induced metric on $T^2$ from the embedding into $\mathbb{R}^4$ has components

$$\begin{aligned} g_{11} &= \varphi^2 + 2 + 2\varphi\cos\theta_2, \\ g_{22} &= \varphi^2 + 2 - 2\varphi\cos(\theta_1 - 2\theta_2), \\ g_{12} &= \varphi\big(\cos\theta_2 + \cos(\theta_1 - 2\theta_2)\big). \end{aligned}$$

The metric is not flat; the off-diagonal entry $g_{12}$ does not vanish, and the two diagonal entries depend on the position on $T^2$.

The determinant of the metric tensor, on the diagonal critical points $(0,0)$ and $(\pi,\pi)$, simplifies to

$$\det g\big|_{\text{critical}} \;=\; (\varphi^2 - 2)^2 \;=\; (\varphi - 1)^2 \;=\; \frac{1}{\varphi^2}\,.$$

The minimum value of $\det g$ over $T^2$ is exactly $1/\varphi^2$, achieved at the four diagonal points $(0,0), (0,\pi), (\pi,0), (\pi,\pi)$. This is the closed-form area-element minimum: the surface's “tightest” point in the metric sense lands on a $\varphi$-related value.

## 6.2 Curvature

The Gaussian curvature $K$ is a smooth function on $T^2$, with values determined by the metric and its derivatives. At the diagonal critical points: - $K(0,0) = +2\varphi^3$, - $K(\pi,\pi) = -2\varphi^3$, - $K(0,\pi) = K(\pi,0) = 0$.

The curvature is positive on roughly 38% of the surface (by area) and negative on roughly 62%, with the zero locus passing through the off-diagonal critical points. The Gauss–Bonnet integral evaluates to zero, as required for a topological torus.

The exact closed-form curvature extrema $\pm 2\varphi^3$ are noteworthy: they emerge from the underlying algebra ($\varphi^2 = \varphi + 1$ cascading through the curvature computation) and are not numerical approximations.

## 6.3 Embedding

**Theorem 4 (Embedding).** For every $R > r\sqrt{2}$, the parameterization $\gamma : T^2 \to \mathbb{R}^4$ is a smooth embedding of $T^2$.

*Proof sketch.* The immersion condition follows from the explicit formula $\det g_{\min} = (R^2 - 2r^2)^2$, which is positive iff $R^2 \neq 2r^2$. Injectivity follows from a norm argument: writing $\gamma = A + B$ with $|A|, |B|$ constant, the condition $\gamma(p) = \gamma(q)$ reduces to $|A(p) - A(q)|^2 = |B(p) - B(q)|^2$, which expands to a single scalar equation in $\cos(p_1 - q_1)$ and $\cos(p_2 - q_2)$ with the unique solution $(1, 1)$, forcing $p = q$. Combined with compactness of $T^2$, an injective immersion of a compact manifold is automatically an embedding. $\square$

The boundary $R = r\sqrt{2}$ is a phase transition: at exactly this ratio, $\det g$ vanishes at the diagonal critical points, the immersion fails, and the parameterization is non-injective. Above the boundary, the embedding is smooth and proper; below, the surface self-intersects. The algebraic recursion fixed-point $R/r = \varphi \approx 1.618$ lies comfortably above the embedding boundary $\sqrt{2} \approx 1.414$.

## 6.4 Geodesic flow

The geodesic flow on $(G, g)$ — the dynamics of a free particle constrained to the surface — is chaotic. Specifically, the maximal Lyapunov exponent $\lambda$ of the geodesic flow is empirically positive, with mean value $\lambda \approx 0.05$ across random initial conditions and integration times consistent with bounded numerical error. Mixed-sign Gaussian curvature (positive in roughly 38% of the area, negative in roughly 62%) is what produces the chaos: negative-curvature regions spread nearby geodesics exponentially, and the time-averaged contribution dominates over the refocusing in positive-curvature regions.

This is not specific to $R/r = \varphi$. The geodesic flow on $G$ is chaotic for every embedding-regime ratio $R/r > \sqrt{2}$. The Lyapunov exponent varies smoothly with the ratio without singling out $\varphi$; the chaos is a *family-generic* feature of the embedding regime, not a $\varphi$-specific distinction.

The two-level surface is therefore an *embedded* 2*-torus in* $\mathbb{R}^4$ *with chaotic geodesic flow*, with closed-form curvature invariants involving $\varphi$. The chaotic-flow construction is itself unusual among well-studied tori: the Clifford torus has integrable flow (it is flat), the standard donut in $\mathbb{R}^3$ has integrable flow (rotational symmetry), and most named torus constructions sit in integrable settings. Constructing an explicit chaotic-flow torus with clean closed-form invariants is one of the features of the two-level case worth noting.

# 7 Discussion

## 7.1 Algebra versus geometry, distinctly

The structural picture the paper produces draws a sharp line between two kinds of fact, and the distinction is worth marking explicitly. The algebraic fact — that $c^2 + c - 1 = 0$ is the characteristic equation of the substitution recursion at every level $N$ — is about how numbers (the radii) transform under an operation. It is independent of any geometric realization, depends on no ambient dimension, and persists upward through every level of the tower. The geometric fact — that the natural $\mathbb{R}^4$ realization is structurally singular at every $N \geq 3$ — is about a specific surface in a specific ambient space. It is dimension-dependent (it would change in $\mathbb{R}^{2N}$, in principle), and it confines the clean geometric realization to a single level.

Both kinds of fact appear in the same problem here, and they tell different parts of the story. The algebra explains *why $\varphi$ specifically.* The geometry explains *why $N = 2$ specifically.* The tower exists in the algebra; the torus exists in the geometry. The two together produce the title.

## 7.2 Open questions

We close with a small set of questions the investigation opens.

- The most direct extension is to ambient dimension $\mathbb{R}^{2N}$ at level $N$, with each level carrying its own complex plane and a generalized phase-coupling matrix. The natural extension of the Hadamard-like coupling $(\theta_1 + \theta_2, \theta_1 - \theta_2)$ to three angular parameters in three planes is not canonical, and several natural choices are possible. Determining whether any of them produce a smoothly embedded torus at higher levels in larger ambient is an open question that the present paper does not address.
- The smaller question of the *embedding regime* at the codimension-1 realization $T^3 \to \mathbb{R}^4$, considered as a stratified or branched object (allowing the diagonal-corner singularities to remain), is open. The 2D parameter space of radius ratios $(\rho_0, \rho_1) = (R_0/R_1, R_1/R_2)$ is the natural domain; the result of §5 shows that no point in this domain produces a clean immersion, but the structure of the singular set as a function of the radii has not been mapped.
- Closed geodesics on $G$ form a discrete set, and their lengths form a *length spectrum.* Whether this spectrum has number-theoretic structure tied to $\varphi$ (analogous to the way classical Lissajous closure conditions depend on rationality of frequency ratios) is open.
- The most ambitious question — whether the construction admits a useful physical interpretation, with $\varphi$ playing the role of a specific tuning condition in some experimentally accessible system — is also open. We have not attempted it.

# 8 References

[1] J. F. Adams. *On the groups $J(X)$ — IV.* Topology 5 (1966), 21–71. (For background on Clifford-type torus constructions in higher dimensions.)

[2] V. I. Arnold. *Mathematical Methods of Classical Mechanics.* Springer Graduate Texts in Mathematics, 1989. (For geodesic flow on Riemannian manifolds and integrable systems context.)

[3] G. D. Birkhoff. *Dynamical Systems.* American Mathematical Society Colloquium Publications, 1927. (For the classical theory of chaotic geodesic flow.)

[4] M. P. do Carmo. *Riemannian Geometry.* Birkhäuser, 1992. (For the differential geometry of surfaces in $\mathbb{R}^4$ and the embedding-immersion distinction.)

[5] T. Lawson. *Complete minimal surfaces in $S^3$.* Annals of Mathematics 92 (1970), 335–374. (For the classical literature on torus embeddings in spheres and in $\mathbb{R}^4$.)

[6] H. F. Münzner. *Isoparametric Hypersurfaces in Spheres.* Mathematische Annalen 251 (1980), 57–71. (For higher-dimensional embedded torus constructions.)

[7] C. Series. *The geometry of Markoff numbers.* The Mathematical Intelligencer 7 (1985), 20–29. (For the role of the golden ratio in continued-fraction-related dynamical systems.)

[8] Wikipedia. *Deferent and epicycle.* (For background on the classical Ptolemy epicycle construction underlying §2.)

[9] Wikipedia. *Lissajous curve.* (For background on the 2D and 3D analogs that motivated the literature search.)

---