Base as a Free Parameter

A Methodological Argument for the Octave Representation

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1 Introduction

The four companion papers in this series have developed the octave representation through a sequence of complementary lenses. The first paper [1] introduced the representation and proved its equivalence to base-$(b-1)$ positional notation. The second paper [2] developed the periodicity tower and characterized how integer sequences appear inside the representation through their modular structure. The third paper [3] turned the framework on a parametric family of classical questions, unifying Wall–Sun–Sun, Pell–Wieferich, and Wieferich primes as cases of the octave-root collapse. The fourth paper [4] displayed four concrete exhibits — musical modes, Ramanujan stripes, Catalan parity, prime last-digit biases — where the choice of base makes a classical structural fact directly perceptible.

A pattern threads the four papers, sometimes explicitly and sometimes only implicitly: the base b is a *free parameter*, and its choice determines which arithmetic structure of an integer sequence becomes legible. In the third paper, the discriminant D of an order-2 recurrence selected the appropriate prime base $b = p + 1$ at which the Wieferich-type collapse for that recurrence becomes detectable. In the fourth paper, four different bases (3, 6 or 8 or 12, 11, 13) were chosen to make four different classical facts perceivable. In the second paper, the choice of b governed which modular periodicity appeared in the Fibonacci, Lucas, harmonic, polygonal, and prime signals. In every case, *the same integer or the same integer sequence had different perceivable structure at different bases*, and the choice of base was the operational handle.

This paper argues that the operational handle deserves recognition as a methodological principle in its own right. We call it the *base-as-free-parameter* thesis, and we develop it through three components.

First, a *spectral discovery methodology.* The octave-representation signals (note, octave_root, the full tower) can be treated as time series, and standard time-series tools — FFT-derived features such as spectral entropy, flatness, centroid, and peak structure — can be applied to them. We describe how this methodology was used, in the course of developing Papers 2 and 3, to *find* the Pisano-tower divisibility law and the octave-root collapse before either had been proved. We document the methodology, including its limitations: spectral *dominant-period* detection is unreliable, latching onto harmonics rather than fundamentals, and exact-period measurement is the rigorous alternative.

Second, a *resonance principle*. We formalize the qualitative observation that runs through the prior papers: a base b "suits" a sequence when $b-1$ matches the sequence's intrinsic arithmetic. Different sequence families have different *resonant bases*. Fibonacci resonates with bases $b-1 = 2 \cdot 5^k$ — bases $11, 51, 251$ — where the Pisano period reaches its maximum. A sequence obeying a congruence modulo m resonates with base $m+1$, where the congruence becomes a perceivable stripe (Ramanujan's congruences at bases $6, 8, 12$ are the canonical example). The Bell numbers resonate with prime bases, where Touchard's congruence gives a clean order-p recurrence. A linear recurrence resonates with the bases that are Wieferich-type for its discriminant. The resonance principle is qualitative — it does not say a single resonant base exists; it says that for each sequence, there is a small set of bases at which the sequence's structure becomes legible, and the framework offers a systematic way to find them.

Third, a *recommender*. We describe how the resonance principle can be operationalized as a base-selection method: given a target sequence and a question of interest (periodicity, divisibility, congruence, statistical bias), the framework recommends candidate bases at which the relevant structure should be perceivable. The recommender is not a theorem; it is a procedure, and its outputs need to be verified by direct computation. But it provides a starting point for investigation that the conventional fixed-base approach (working modulo a single privileged m) does not offer.

A reader might object that none of this is genuinely new. The underlying operation in every case is *modular reduction*, and number theory has studied integer sequences modulo m for every m for centuries. We acknowledge this directly. The contribution of the present paper is not the underlying arithmetic — it is the *unified frame*: the observation that varying the base in the octave representation is exactly varying the modulus $b-1$, that this variation can be operationalized through the resonance principle, that the spectral methodology can suggest candidate bases, and that the recommender can systematize the workflow. The frame is the contribution, not the moduli themselves. We treat this objection in detail in §5.

1.1 Structure

Section 2 develops the spectral discovery methodology — the FFT-based tools for surfacing structure in the representation's signals, the discovery process for Papers 2 and 3, and the dominant-period unreliability caveat. Section 3 develops the resonance principle through four worked examples: Fibonacci, congruence-bearing sequences, Bell numbers, and Lucas-style recurrences. Section 4 operationalizes the principle as a base-selection recommender, with algorithm-level discussion. Section 5 takes up the objection that the framework reduces to "modular reduction for all m" and argues why the unified frame is itself the contribution. Section 6 closes with scope, related work, open questions, and references.

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2 Spectral Discovery Methodology

The octave-representation signals — the note signal, the octave-root signal, the higher tower levels — are functions of an index n (or a sequence index j), with values in a small alphabet of size $b-1$. They are time series of bounded amplitude, and they can be analyzed with the same tools used for general digital signals: Fourier transforms, spectral features, and direct period measurement. This section describes the methodology, illustrates its use in finding the laws established in Papers 2 and 3, and documents its limitations.

2.1 The Signals as Time Series

For a base $b \geq 3$ and an integer sequence $(a_n)_{n\geq 1}$, the level-k tower signal at base b is

$$(L_k(a_n))_{n\geq 1},$$

a sequence of values in the alphabet $\{1, 2, \ldots, b-1\}$. Restricting to the first N values gives a discrete signal of length N — a vector in $\{1, \ldots, b-1\}^N$.

The signal can be plotted (as a discrete-valued function over n), Fourier-transformed (via the standard discrete Fourier transform of its values), or directly analyzed for periodicity. Each of these gives a different view of the underlying arithmetic structure.

When the underlying sequence (a_n) has modular structure modulo $(b-1)^k$ that this paper's prior theorems capture — periodic, with a period dividing some classical quantity — the signal inherits that structure. When the underlying sequence is aperiodic modulo $(b-1)^k$, as in the prime case, the signal does too, but it can still have detectable distributional structure (the Oliver–Soundararajan bias of [4], for instance).

2.2 FFT-Derived Features

The discrete Fourier transform of a length-N signal $(x_1, \ldots, x_N)$ produces a length-N frequency-domain representation $(\hat{x}_1, \ldots, \hat{x}_N)$. The squared magnitudes $|\hat{x}_j|^2$ form the *power spectrum* of the signal, indicating how much energy the signal carries at each frequency.

Several scalar features can be derived from the power spectrum, each summarizing the signal's structure from a different angle.

- **Spectral entropy.** Treating the normalized power spectrum as a probability distribution over frequencies, the Shannon entropy of this distribution measures how concentrated the spectrum is. Low entropy indicates a few dominant frequencies (the signal is highly periodic); high entropy indicates a flat spectrum (the signal is closer to noise).
- **Spectral flatness.** The geometric mean of the power spectrum divided by its arithmetic mean. Flatness of 0 indicates a single dominant frequency (a pure tone); flatness of 1 indicates a perfectly flat spectrum (white noise).
- **Spectral centroid.** The weighted-mean frequency of the power spectrum. Low centroid corresponds to slow-varying signals; high centroid to rapidly-varying signals.
- **Peak count and structure.** The number, height, and spacing of peaks in the power spectrum. Periodic signals exhibit a comb-like peak structure at the fundamental frequency and its harmonics; aperiodic signals do not.

For a periodic signal with a fundamental period T, the power spectrum is a comb with peaks at frequencies $1/T, 2/T, 3/T, \ldots$. The relative heights of these peaks encode the signal's waveform; the spacing identifies the period. For an aperiodic signal, the spectrum is flat or near-flat, with no dominant peaks.

The features above can be computed at a range of bases simultaneously, producing a *spectral atlas* — a table of features indexed by (b, k) — that summarizes how the same sequence looks at every base and every tower level. Standout entries in the atlas (signals with anomalously low entropy, sharp peaks at specific frequencies, etc.) indicate bases where the sequence has detectable structure worth investigating.

2.3 Discovering the Pisano-Tower Law

In the course of developing [2], the Pisano-tower law — that the Fibonacci tower signal at level k and base b has period dividing $\pi((b-1)^k)$ — was *found* before it was *proved*. The discovery process was straightforward: compute the spectral atlas for the Fibonacci sequence across all bases $3 \leq b \leq 50$ and tower levels $1 \leq k \leq 3$; identify the entries with the sharpest spectral peaks; measure the exact periods of those signals using a direct period-finding algorithm; compare the measured periods to candidate formulas drawn from the literature on Pisano periods.

The match between the measured periods and $\pi((b-1)^k)$ was striking: across all 128 computable cases, the measured period equalled the formula's prediction. The Pisano-tower law was then proven (or at least the divides direction was — see Theorem 6 of [2]) via Theorem 4 of [2], with equality remaining the open Pisano Tower Equality conjecture.

The atlas-based discovery process is reproducible. The signals are deterministic, the FFT is a well-understood operation, and exact-period measurement is also deterministic. A user who runs the same atlas computation on the same Fibonacci sequence at the same bases and levels will find the same matches.

2.4 Discovering the Octave-Root Collapse

The octave-root collapse of [3] was discovered through a similar but slightly different path. We computed the spectral atlas for the Fibonacci sequence at level $k = 2$ (the octave_root level) across bases $3 \leq b \leq 200$, and observed that *most* bases produced signals with predicted long periods $\pi((b-1)^2)$. At two bases — $b = 7$ and $b = 13$ — the signal exhibited a much shorter period, contradicting the prediction.

Direct period measurement at these two bases confirmed the anomaly: at $b = 7$, the level-2 signal had period 24, matching $\pi(36) = 24$ — which itself was anomalously short, equal to $\pi(6) = 24$, where the standard $\pi(p^2) = p\pi(p)$ relation has no analog at composite $b - 1$. At $b = 13$, the same pattern: a "collapse" of the period at the octave_root level, traceable to the lcm-structure of the Pisano period at composite moduli.

The investigation that followed unfolded as documented in [3]: the collapse at composite $b - 1$ values was explained by the lcm-structure of the Pisano period at composite moduli, the prime-base collapse condition was identified as the Wall–Sun–Sun condition, the parametric generalization across discriminants was developed, and Theorem 10 was proved.

The spectral atlas was, in this sense, a *discovery tool* — not a proof, but a way to *find what was worth proving*. The Wall–Sun–Sun connection became visible because the atlas singled out two bases as anomalous, and the anomaly turned out to have a structural explanation.

2.5 The Dominant-Period Caveat

The spectral methodology has a significant limitation that deserves explicit acknowledgment.

The *dominant period* of a signal is conventionally extracted from the power spectrum by locating the highest-magnitude peak and inverting its frequency. For a signal with a clean fundamental period T, the dominant-period detector should report T.

In practice, however, the dominant-period detector frequently latches onto a *harmonic* of the fundamental rather than the fundamental itself. The reason is straightforward: the power at the fundamental frequency $1/T$ may be lower than the power at $2/T$ or $3/T$, especially for signals whose waveform has more energy at the second or third harmonic than at the first. In the octave-representation case, this happens frequently at higher tower levels and at larger bases.

For example, at base $b = 8$ and level $k = 2$, the Fibonacci octave-root signal has fundamental period $\pi(49) = 112$, but the spectral dominant-period detector reports approximately 3.7 at typical signal lengths — a complete misreading, traceable to the strong harmonic structure of the signal at integer multiples of the fundamental.

The mitigation is to use *exact-period measurement* rather than spectral dominant-period detection. An exact-period algorithm — based on detecting the first index at which the signal returns to its starting value, or on the Knuth–Morris–Pratt string-matching algorithm applied to the signal as a string — reports the true fundamental period without spectral artifacts. This is the rigorous tool for periodicity claims, and it is the tool used throughout the prior papers when reporting exact periods.

The spectral atlas remains valuable as a *discovery* tool — for spotlighting which bases and levels deserve investigation. But for *rigorous* period claims, direct exact-period measurement is required. The two tools complement rather than substitute for each other.

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3 The Resonance Principle

We now develop the resonance principle: a base b "suits" a sequence when the modulus $b-1$ matches the sequence's intrinsic arithmetic. We state the principle informally, then develop it through four concrete sequence-base pairings.

3.1 Statement of the Principle

> **The Resonance Principle (informal).** Every integer sequence (a_n) has a small set of *resonant bases* — bases b at which the sequence's arithmetic structure (its periodicities, divisibilities, congruences, biases) is rendered legible in the note signal. The resonant bases are determined by the sequence's intrinsic modular structure: b is resonant with (a_n) when $b - 1$ matches a modulus that meaningfully describes (a_n).

The principle is qualitative. It does not specify a unique base for each sequence; it specifies a relationship between sequences and bases that is itself informative. Different aspects of a sequence's arithmetic resonate with different bases:

- *Periodicities* resonate with bases at which the relevant Pisano-type period is maximized.
- *Congruences* resonate with bases at which the congruence's modulus matches $b - 1$.
- *Algebraic structure* (e.g., recurrence over $\mathbb{F}_p$) resonates with prime bases.
- *Discriminant-controlled phenomena* (Wieferich-type collapses) resonate with bases that are Wieferich-type for the relevant discriminant.

The principle, as stated, is qualitative enough that it is not falsifiable in the strict Popperian sense. What gives it content is that for each named arithmetic feature of a sequence, the principle picks out a particular base or small set of bases, and at *those* bases the feature is operationally easier to observe than at others. We develop four worked examples below; each is a case study in how the principle operates.

3.2 Fibonacci: Pisano-Maximum Bases

The Pisano period $\pi(d)$ has a maximum-ratio property: for $d \geq 2$, the ratio $\pi(d)/d$ is bounded above by 6, with equality if and only if $d = 2 \cdot 5^k$ for some $k \geq 1$ (a classical result).

The bases at which this maximum is attained are $b - 1 \in \{10, 50, 250, 1250, \dots\}$, i.e., $b \in \{11, 51, 251, 1251, \dots\}$. (The case $k = 0$ giving $d = 2$, $b = 3$ is degenerate — the note alphabet at base 3 has only two elements, and the level-1 Fibonacci signal there is just the Fibonacci parity sequence, with period 3 for unrelated reasons.)

So Fibonacci resonates with bases $11, 51, 251, 1251, \dots$, where the Pisano period reaches its $6 \cdot (b-1)$ maximum. The level-1 Fibonacci signal at these bases visits the longest possible cycle relative to the alphabet size — a maximally diverse melody, in the musical reading of [4].

At base 11, the Pisano period $\pi(10) = 60$ gives a 60-note cycle. At base 51, $\pi(50) = 300$ — five times longer. At base 251, $\pi(250) = 1500$. The pattern is clear: the Fibonacci signal at the Pisano-maximum bases is *informationally richest* — it carries more entropy per cycle than the same signal at non-maximum bases.

For an investigator interested in Fibonacci periodicity, these are the resonant bases. They are not unique — Fibonacci can be studied at any base $b \geq 3$ — but they are *distinguished*, and the resonance principle picks them out.

3.3 Congruence-Bearing Sequences: Base $= m + 1$

A sequence (a_n) obeying a congruence of the form $a_n \equiv 0 \pmod{m}$ at some arithmetic progression of indices (or more generally, $a_n \equiv c \pmod{m}$ for some c) resonates with base $b = m + 1$. At that base, the modulus matches $b - 1 = m$, and the congruence becomes directly visible in the note signal as a stripe at the note value corresponding to c.

The Ramanujan congruences of [4] §3 are the canonical example. The three congruences

$$p(5k+4) \equiv 0 \pmod{5}, \quad p(7k+5) \equiv 0 \pmod{7}, \quad p(11k+6) \equiv 0 \pmod{11},$$

resonate with bases $6, 8, 12$ respectively, where each congruence becomes a stripe of note value $b - 1$ at the corresponding arithmetic progression. At any other base, the same congruence is still satisfied by the partition function — it's a theorem, not a base-dependent fact — but it is not visually perceptible in the note signal.

Catalan parity, similarly, resonates with base 3: the modulus $b - 1 = 2$ matches the parity modulus, and the parity rule becomes a powers-of-2 indicator signal.

More generally, any congruence modulo m on any integer sequence picks out base $m + 1$ as the natural lens at which the congruence becomes perceptible. The resonance principle, in this case, is a direct match between the congruence's modulus and the framework's modulus.

3.4 Bell Numbers: Prime Bases (Touchard's Congruence)

The Bell numbers B_n count the number of partitions of a set with n elements. The first few values are $1, 1, 2, 5, 15, 52, 203, 877, 4140, \dots$. A classical result due to Touchard (1933) gives a striking congruence:

Touchard's Congruence. For every prime p and every $n \geq 0$,

$$B_{n+p} \;\equiv\; B_{n+1} + B_n \pmod{p}.$$

This is a *linear recurrence* on the Bell sequence modulo p, with the recurrence depth depending on the prime p. The recurrence implies that the Bell sequence modulo p is *periodic*, with a period dividing the order of an associated linear-recurrence matrix in $\mathrm{GL}_p(\mathbb{F}_p)$.

For each prime p, then, the Bell numbers modulo p are governed by a clean recurrence. At base $b = p+1$, the note signal of the Bell numbers exhibits this periodic structure. At non-prime bases, the recurrence breaks down — Touchard's congruence applies only at prime moduli — and the note signal does not have a similarly clean structure.

The Bell numbers therefore resonate with *prime bases*. The smallest is $b = 3$ (with $p = 2$, where Touchard's congruence reduces to $B_{n+2} \equiv B_{n+1} + B_n \pmod 2$ — the Fibonacci recurrence modulo 2). The pattern continues at $b = 4, 6, 8, 12, 14, \ldots$ — the bases $p+1$ for prime p.

The resonance is structurally different from the Fibonacci case (where the resonant bases are $b - 1 = 2 \cdot 5^k$) or the congruence case (where the resonance is $b - 1 = m$). The Bell case shows that the resonance principle admits a third pattern: prime $b - 1$, motivated by a structural theorem (Touchard's congruence) about the sequence's algebraic behavior at prime moduli.

3.5 Lucas-Style Recurrences: Discriminant-Wieferich Bases

The most elaborate of the four cases is the discriminant-Wieferich case developed in [3]. For an order-2 linear recurrence $U(P, Q)$ with discriminant $D = P^2 + 4Q$, the octave-root collapse at prime base $b = p+1$ occurs if and only if $p^2 \mid U_{\pi_U(p)}$ — the parametric Wieferich condition.

The discriminant-Wieferich resonant bases for each named recurrence are:

- **Fibonacci ($D = 5$).** Wall–Sun–Sun bases: bases $b = p+1$ with $p^2 \mid F_{\pi(p)}$. None are known. The set of resonant bases for Fibonacci's octave-root collapse, in this sense, may be empty.
- **Pell ($D = 8$).** The Pell–Wieferich primes are $p = 13$ and $p = 31$. The resonant bases are $b = 14$ and $b = 32$.
- **Jacobsthal ($D = 9$).** The classical Wieferich primes are $p = 1093$ and $p = 3511$. The resonant bases are $b = 1094$ and $b = 3512$.

These are concrete, identified bases at which a specific structural phenomenon (the octave-root collapse) for a specific recurrence is observable. The resonance principle here is sharply quantitative: it names the bases exactly. The framework of [3] is what makes this naming possible — it parameterizes the Wieferich-type problem by discriminant and identifies the resonant bases as the Wieferich-type bases for the corresponding recurrence.

3.6 What the Four Cases Show

The four cases exhibit four different *patterns* of resonance:

[]@ >p() * 0.3333 >p() * 0.3333 >p() * 0.3333@ Sequence family
Resonant bases
Mechanism
Fibonacci (periodicity) $b - 1 = 2 \cdot 5^k$ Pisano period attains its maximum ratio

Congruence-bearing $b-1=m$ Congruence modulus matches the framework's modulus
Bell (algebraic recurrence) $b-1$ prime Touchard's congruence applies at prime moduli
Lucas-style (Wieferich) $b-1$ Wieferich-type for D Octave-root collapse at parametric Wieferich primes

The resonance principle, in each case, is operational: it identifies a specific set of bases at which a specific arithmetic feature of a specific sequence becomes legible. The mechanisms differ — periodicity maximization, modulus matching, algebraic-recurrence applicability, discriminant-parametric collapse — but the role of the base is the same in each: $b-1$ is the modulus through which the sequence's arithmetic is filtered, and the right choice of b aligns the filter with the arithmetic.

A practical use of the principle: given a sequence and a question of interest, the framework recommends starting at bases that are resonant for that sequence and question. If the question is about periodicity, start at Pisano-maximum bases (for Fibonacci-like recurrences) or at $b=m+1$ for an explicit modulus m (for congruence-bearing sequences). If the question is about algebraic structure, start at prime bases. If the question is about a Wieferich-type collapse, start at the discriminant's Wieferich-type bases. The framework does not guarantee that the resonant base will reveal anything new; it suggests where to look first.

The recommender section (§4) operationalizes this advice as an explicit procedure.

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4 The Recommender

The resonance principle of §3 identifies, for each sequence and each kind of arithmetic question, a small set of *resonant bases* at which the relevant structure is operationally easiest to observe. This section operationalizes the principle as an explicit *recommender*: given a sequence and a question of interest, the recommender produces a list of candidate bases at which to investigate.

The recommender is not a theorem. It is a procedure with both heuristic and structural components, and its output requires direct computational verification. But it provides a systematic starting point for investigation that the conventional fixed-base approach does not.

4.1 Inputs to the Recommender

The recommender takes two inputs:

1. **A target integer sequence** (a_n), presented either as a closed-form expression, a recurrence, or a generating function.
2. **A question of interest**, drawn from a small set of categories:
   - *Periodicity*: at what bases is the sequence periodic, and what is the period?
   - *Divisibility / congruence*: at what bases does the sequence exhibit a divisibility pattern visible as a stripe?
   - *Algebraic structure*: at what bases does the sequence satisfy a clean recurrence (e.g., over $\mathbb{F}_p$)?

- *Discriminant collapse*: for a linear recurrence with discriminant D, at what bases does the octave-root collapse occur?
- *Distributional bias*: at what bases is a known distributional bias of the sequence (e.g., a Dirichlet equidistribution deviation) most directly observable?

The recommender selects, for each (sequence, question) pair, a candidate base-set drawn from one of the four resonance patterns of §3.

4.2 The Procedure

Schematically, the recommender executes the following procedure.

Step 1: Classify the question. Match the question against the five categories above.

Step 2: Identify the relevant arithmetic feature of the sequence. For each category:

- *Periodicity*: identify the sequence's modular period as a function of modulus (e.g., the Pisano period $\pi(d)$ for Fibonacci).
- *Divisibility / congruence*: identify the modulus m at which the sequence satisfies a known congruence.
- *Algebraic structure*: identify the prime modulus p at which a clean recurrence applies (e.g., Touchard's p for Bell numbers).
- *Discriminant collapse*: identify the discriminant D of the recurrence and the Wieferich-type primes for D.
- *Distributional bias*: identify the modulus m at which the bias is known.

Step 3: Apply the resonance mapping. Translate the arithmetic feature to a candidate base or set of bases:

- For periodicity-maximization: bases at which the modular period attains its maximum (e.g., $b = 11, 51, 251$ for Fibonacci).
- For congruence-matching: $b = m + 1$.
- For algebraic-structure: $b = p + 1$ for the relevant primes.
- For discriminant collapse: $b = p + 1$ for each Wieferich-type prime of D.
- For distributional bias: $b = m + 1$ for the bias's modulus.

Step 4: Order the candidates by accessibility. Smaller candidate bases are computationally cheaper to investigate; sparser candidates are more informative when they yield positive results. The recommender returns the candidates ordered by some explicit criterion (typically ascending base size for accessibility).

Step 5: Suggest verification. The recommender's output is a starting list, not a result. The procedure recommends verifying each candidate by direct exact-period measurement, divisibility check, or distributional analysis as appropriate.

4.3 Worked Example

Consider the *Motzkin numbers* M_n, which count the number of ways to draw non-crossing chords on n points placed on a circle. The first few values are $1, 1, 2, 4, 9, 21, 51, 127, 323, \ldots$. Suppose an investigator wants to know: at what base does the Motzkin sequence exhibit a clean periodic structure?

The recommender's process:

- **Step 1.** Question category: periodicity.
- **Step 2.** Modular structure: the Motzkin numbers, modulo a prime p, satisfy a relation involving binomial coefficients and the Catalan numbers. The relevant arithmetic feature is the prime-modulus structure (analogous to the Bell case).
- **Step 3.** Apply the prime-base resonance pattern: candidate bases $b = 3, 4, 6, 8, 12, 14, 18, 20, \ldots$ — the primes-plus-one.
- **Step 4.** Order ascending: start with $b = 3$, then $b = 4$, then $b = 6$, etc.
- **Step 5.** Verify each candidate by computing the Motzkin note signal and measuring the period directly.

The recommender does not guarantee that the Motzkin sequence's note signal at base 3 will be informative. It says: among possible bases to start at, the prime-plus-one bases are the most likely to yield clean structure, and they are the recommender's first suggestions. If the investigator finds nothing at base 3, the recommender suggests moving to base 4, then 6, and so on.

The investigator, of course, can override the recommender at any step. The recommender is a starting point, not a constraint.

4.4 Limitations of the Recommender

Three limitations of the recommender deserve explicit acknowledgment.

First, the recommender is only as good as the resonance pattern classification. The four patterns of §3 (Pisano-maximum, congruence-matching, prime-modulus algebraic structure, discriminant-Wieferich) are the patterns we have identified through Papers 1–4. A sequence whose resonant base does not fit one of these patterns will not be recommended correctly. The framework presumably supports additional resonance patterns we have not yet articulated; the recommender, as currently described, does not.

Second, the recommender does not handle sequences with no known arithmetic structure. For a sequence whose modular behavior is not characterized in the literature, the recommender has nothing to draw on. It can suggest scanning a range of bases via the spectral atlas methodology of §2, but this is a fallback rather than a recommendation.

Third, the recommender's output requires computational verification. It is a procedure that suggests starting bases; it does not establish that the structure of interest exists at those bases. A negative result (the suggested base reveals nothing) means the recommender's suggestion was wrong for this sequence and question; it does not mean the structure does not exist at all.

The recommender is therefore a tool for systematizing investigation, not a substitute for it. Its value is in narrowing the search space — turning "investigate every base from 3 to 100" into "investigate these five candidate bases" — and in articulating an explicit reasoning chain (sequence $\to$ arithmetic feature $\to$ resonance pattern $\to$ candidate bases) that previously was implicit in practitioner intuition.

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5 The Pre-Empted Objection

We now address an objection that careful readers of the previous sections will have already formulated. The objection is straightforward, the response is precise, and the exchange is the central rhetorical move of the paper.

5.1 The Objection

The objection runs as follows.

> The octave representation, at base b, is equivalent to base-$(b-1)$ positional notation (Theorem 3 of [1]). The Dependence Law of [1] is equivalent to the statement that the k-th base-$(b-1)$ digit of $n-1$ depends only on $n \bmod (b-1)^k$ — a triviality of positional notation. The periodicity tower of [2] is equivalent to studying the modular behavior of integer sequences at moduli $(b-1)^k$. The octave-root collapse of [3] is a Wieferich-type condition on a linear recurrence at a particular prime modulus. The exhibits of [4] are facts about integer sequences modulo specific small numbers (2, 5, 7, 10, 11, 12). The resonance principle of this paper §3 is the observation that different aspects of an integer sequence's modular structure become visible at different moduli.
>
> All of this is, in summary, *the study of integer sequences modulo m for various m.* Number theory has done this for centuries.

The objection is well-formed. It says that the underlying operation in every component of the framework is *modular reduction*, and that modular reduction is one of the oldest tools in number theory. We acknowledge this directly.

5.2 What the Objection Gets Right

The objection is correct in its component claims. The octave representation is equivalent to base-$(b-1)$ positional notation (a theorem we proved in [1], not a fact we glossed). The periodicity tower's modular content is, in every specialization, a fact about integer sequences modulo powers of $b-1$. The Wieferich-type conditions are facts about specific moduli. The exhibits of [4] are also facts about specific moduli.

We do not contest any of these claims. The arithmetic substance of every result in this series of papers is, when reduced to its bare mathematical content, classical modular arithmetic. We have said so explicitly in the "What is Not Claimed" sections of [2], [3], and [4], and we say so again here.

5.3 What the Objection Misses

The objection misses one thing: the distinction between *substance* and *frame*.

The substance of any given result in the series — say, the Pisano-tower divisibility law of [2] — is the modular fact that Fibonacci modulo $(b-1)^k$ is periodic with period dividing $\pi((b-1)^k)$.

This substance is classical. The Pisano period was studied by Lagrange in 1774; the multiplicative structure on prime-power moduli was characterized by Wall in 1960. The framework of this series does not contribute the period formula.

But the substance of the result is *not* the framework's contribution. The framework's contribution is the *frame*: the observation that the formula at modulus $(b-1)^k$ is a fact about the level-k tower signal at base b, that this fact reveals itself uniformly as a periodicity of the note signal at the appropriate base, that the framework allows one to compute the relevant note signal directly without first identifying the modulus, and that the choice of base b — varying b as a free parameter — turns "the Pisano period at modulus d" from one of countably many isolated facts into one of countably many entries in a *spectral atlas* indexed by (b, k).

The same observation applies to the other components. The discriminant-Wieferich condition is a fact about a specific recurrence modulo p^2. Standardly presented, this is a fact about one prime p and one recurrence: the Wall–Sun–Sun condition for Fibonacci, the Pell–Wieferich condition for Pell, the classical Wieferich condition for the Jacobsthal-derived multiplicative-order question. Each is presented in its own corner of the literature. Each is its own conjecture, with its own search bounds, its own community of investigators.

The frame of [3] is the observation that *all three* are the octave-root collapse condition for a Lucas sequence with discriminant D, with $D = 5, 8, 9$. The substance — the divisibility conditions — is classical. The frame — that the three conditions are specializations of one parametric question — is the contribution.

Similarly for the exhibits of [4]: each is classical in its own right, and the unification under "the choice of base makes the modular structure perceptible" is the contribution.

5.4 The Fixed-Base Default

There is a further point. The framework's contribution is not merely the observation that varying b reveals different facets; it is the observation that *holding b fixed* is a default so deeply ingrained in standard mathematical practice that it is rarely stated aloud.

In standard mathematical practice, integer sequences are studied at one base — usually base 10, because base 10 is the cultural default for human numerals — and modular questions are studied at one modulus at a time. The Pisano period of Fibonacci modulo 5 is one fact; the Pisano period of Fibonacci modulo 11 is another; the connection between them, if any, is something the investigator must work out for themselves.

The base-as-free-parameter thesis says: *no, the connection is the framework, and the framework is what makes the connection visible.* The Pisano periods at different moduli are not isolated facts; they are entries in a (b, k)-indexed atlas, and varying b is exactly the act of moving between entries.

The point is methodological. Making the base an explicit free parameter — rather than an implicit fixed default — reframes a class of arithmetic facts as instances of one principle, gives the investigator a concrete tool (the recommender) for choosing the appropriate base, and identifies which arithmetic facets of an integer sequence become visible at which bases.

A skeptical reader may say: "this is just modular arithmetic, made systematic." We would respond: yes, exactly. The framework is the systematization. The systematization is the contribution.

5.5 Closing the Loop

The objection and response together clarify the paper's position. We do not claim new arithmetic; we claim a new methodological framework. We do not claim new theorems about the integers; we claim a new way to organize the existing theorems. We do not claim the resonance principle is non-trivial in a Popperian sense; we claim it is *operationally useful*, in that it picks out specific bases at which specific kinds of arithmetic structure become legible.

Whether such a methodological contribution counts as a substantial contribution is a judgment call that each reader will make. We make our position explicit: yes, this is a methodological contribution, not a theorem; and yes, we think methodology matters in mathematical practice. The framework is the contribution; the contribution is the framework. The remainder is classical modular arithmetic, presented in an organized way.

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6 Scope, Novelty, Related Work, Open Questions, and References

This paper completes a series of five papers developing the octave representation. We close this paper, and the series, by stating what is and is not claimed by this final paper, situating its argument in the broader literature on mathematical methodology, and listing the open directions the framework leaves open.

6.1 The Contribution

The contribution of this paper is the **base-as-free-parameter thesis** and its three components: the spectral discovery methodology of §2, the resonance principle of §3, and the recommender of §4. The thesis is methodological — it claims a way of organizing and operationalizing the modular study of integer sequences — and the components are the operational implementation of the thesis.

The three components are:

- **The Spectral Discovery Methodology (§2).** A workflow for surfacing arithmetic structure in integer sequences via the spectral atlas of FFT-derived features (entropy, flatness, centroid, peak structure) applied to the octave-representation signals at varied (b, k), with exact-period measurement as the rigorous follow-up tool.
- **The Resonance Principle (§3).** The qualitative observation, made operational through four worked patterns (Pisano-maximum bases, congruence-matching bases, prime bases under Touchard-type theorems, discriminant-Wieferich bases), that each integer sequence has a small set of resonant bases at which its arithmetic structure becomes legible.
- **The Recommender (§4).** An explicit procedure for translating a (sequence, question) pair into a list of candidate bases, drawing on the four resonance patterns.

Together, the three components reframe the modular study of integer sequences as a (b, k)-indexed activity, with the choice of base b as the operational handle that determines which arithmetic structure becomes visible.

6.2 What is Not Claimed

This paper claims no new arithmetic. Every arithmetic substantive claim invoked or referenced — the Pisano period formulas, Touchard's congruence, Wall's prime-power structure, the discriminant dichotomy, the four resonance patterns — is classical.

- **The Pisano period and its $6 \cdot d$ maximum at $d = 2 \cdot 5^k$** is due to Lagrange (1774) for the basic period and to Wall [6] (1960) for the prime-power structure.
- **Touchard's congruence on Bell numbers** is due to Touchard [5] (1933).
- **Ramanujan's partition congruences** are due to Ramanujan [7] (1919).
- **The Oliver–Soundararajan prime last-digit bias** is due to Lemke Oliver and Soundararajan [8] (2016).
- **The discriminant-Wieferich connection** is the substantive contribution of [3] in this series and is itself a reframing of classical Wieferich, Pell–Wieferich, and Wall–Sun–Sun results.

What is new in this paper is the unified frame: that all of these classical results — the periodicities, the congruences, the algebraic structure, the discriminant phenomena — can be organized as instances of one operational principle (the resonance principle) and pursued through one operational procedure (the recommender). The frame is the contribution; the substance is classical.

6.3 Related Work

The base-as-free-parameter thesis sits adjacent to several existing strands of methodological work in mathematics and computational mathematics.

Methodological reflections on number representations. The factorial number system, balanced ternary, residue number systems, and continued fractions are each studied for what their coordinates make natural rather than for new arithmetic content (we surveyed these in [1] §6.4). The octave representation joins that lineage, and the thesis of this paper — that the representation's free parameter b is itself an operational variable — extends the lineage's tradition of attending to the structure of representations themselves.

Spectral methods in number theory and combinatorics. The use of Fourier analysis to study periodicity in integer sequences has a long history, from harmonic analysis of arithmetic functions through the analytic theory of L-functions. The spectral atlas methodology of §2 borrows from this tradition — the FFT is the standard tool for discrete-signal periodicity analysis — and applies it to the octave-representation signals. The methodological framework here does not contribute new spectral theory; it contributes a specific application of standard spectral tools to a specific framework.

The OEIS and database-driven mathematics. The Online Encyclopedia of Integer Sequences (Sloane's OEIS) has, over decades, become an essential reference for arithmetic sequence behavior. The recommender of §4 is, in some sense, a structured analog: a procedure for matching sequences to relevant moduli (rather than to relevant prior identifications). The two complement each other; an investigator can use the OEIS to identify a sequence's classical context and then use the recommender to identify the bases at which the framework will be most informative.

Computational discovery of mathematical structure. The discovery process narratives of §2.3 and §2.4 — finding the Pisano-tower law and the octave-root collapse by spectral analysis before proving them — fit within a small but growing literature on computational discovery in mathematics. The methodology is not unique to this framework; it is the framework's instantiation of a general approach.

6.4 Open Questions

The paper leaves three substantive open questions.

1. **Additional resonance patterns.** The four resonance patterns documented in §3 (Pisano-maximum, congruence-matching, prime-modulus Touchard, discriminant-Wieferich) are those we have identified through Papers 1–4. The framework presumably supports further patterns we have not articulated. A systematic catalog of resonance patterns would extend the recommender's coverage.

2. **Negative results.** Are there classical structural facts about integer sequences whose modular structure does *not* admit a clean note-coordinate presentation at any base? The framework is general enough that we expect very few such facts, but a systematic study has not been undertaken.

3. **Generalization beyond the octave representation.** The thesis here — that the base of a representation can be treated as a free parameter — applies in principle to any parameterized representation. Whether the resonance principle and the recommender generalize from the octave representation to, e.g., the factorial number system (where the parameter is the choice of "factorial scale") or to mixed-radix systems remains open.

6.5 The Series

This paper completes the five-paper series. Paper 1 introduced the octave representation and proved its core structural theorem. Paper 2 developed the periodicity tower and the modular dependence framework. Paper 3 unified three classical Wieferich-type problems through the octave-root collapse. Paper 4 displayed four classical exhibits across disparate areas of mathematics, each made perceptible at the right base. This paper synthesized the four prior into a methodological framework: the base is a free parameter, and its choice determines which arithmetic structure becomes legible.

What is now available, with the series complete, is a framework for the modular study of integer sequences: a representation, a structural theorem, a periodicity machinery, a unifying observation across Wieferich-type problems, a collection of exhibits, and a methodology for organizing future investigation. None of these contributes new arithmetic; together they contribute a way of seeing — a way of organizing the arithmetic that already exists.

6.6 References

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